Stationary stochastic processes

Maria Sandsten

Lecture 9

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Last week: Differentiation

Let $X(t)$, $t \in T$, be a weakly (and Gaussian) stationary process with covariance function $r_X(\tau)$. Then the following statements are equivalent:

- $X(t)$ is differentiable in quadratic mean
- $r_X(\tau)$ is two times differentiable for every τ
- $\int_{-\infty}^{\infty} (2\pi f)^2 R_X(f) df < \infty$

Last week: Differentiation

The derivative $X'(t)$, $t \in T$, is weakly (and Gaussian) stationary with expected value,

$$m_{X'} = 0,$$

covariance function,

$$r_{X'}(\tau) = -r_X''(\tau),$$

spectral density,

$$R_{X'}(f) = (2\pi f)^2 R_X(f).$$

The cross-covariance function between $X(t)$ and $X'(t)$ is

$$r_{X,X'}(\tau) = r_X'(\tau).$$

Last week: Cross-spectrum

The cross-spectrum is complex valued,

$$R_{X,Y}(f) = A_{X,Y}(f)e^{i\Phi_{X,Y}(f)},$$

where $A_{X,Y}(f) = |R_{X,Y}(f)|$ is the cross-amplitude spectrum and $\Phi_{X,Y}(f) = \arg R_{X,Y}(f)$ is the phase spectrum.

The (squared) coherence spectrum is defined as

$$\kappa^2_{X,Y}(f) = \frac{|R_{X,Y}(f)|^2}{R_X(f)R_Y(f)},$$

and $0 \leq \kappa^2_{X,Y} \leq 1$.

Schedule for today

- Chapter 7.2.2: The Moving Average process, the MA-process
- Chapter 7.2.1: The Auto-Regressive process, the AR-process
- Poles and zeros, pp 176-179
- Chapter 7.2.3: The mixed model, the ARMA-process

Modeling of time series

There are many good reasons for using discrete time models:

- Many series are actually generated in a feedback system, i.e. Auto-Regressive (AR) models.
- The discrete time models are simple and a smart choice of parameters can approximate many covariance and spectrum structures.
- Parameter estimation is simple.
- Especially AR-models are easy to use in prediction.

However, if the model is wrong, both bias and variance of the estimate could be very large.

The MA(q)-process

A Moving Average (MA) process of order q, referred to as an MA(q)-process, is given by

$$X_t = c_0 e_t + c_1 e_{t-1} + \ldots + c_q e_{t-q},$$

where e_t, $0, \pm 1, \pm 2, \ldots$, is a zero-mean white noise with variance σ^2. The expected value is

$$E[X_t] = 0.$$

The covariance function is given from

$$r_X(\tau) = C[X_t, X_{t+\tau}].$$

The spectral density is

$$R_X(f) = r_X(0) + 2\sum_{\tau=1}^{q} r_X(\tau)\cos(2\pi f \tau), \quad -1/2 < f \leq 1/2.$$

For examples see page 32 of Lecture 1 and pages 17-18 of Lecture 4.

The AR(p)-process

An Auto-Regressive (AR) process of order p, referred to as an AR(p)-process, is given by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t,$$

where, e_t, $t = 0, \pm 1, \pm 2 \ldots$, is a zero-mean white noise with variance σ^2. The expected value is

$$E[X_t] = 0.$$

Pseudoproof: $E[X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p}] = E[e_t] = 0$, giving

$$(1 + a_1 + a_2 + \ldots + a_p) m_X = 0.$$

It can be shown that $1 + a_1 + a_2 + \ldots + a_p \neq 0$, therefore $m_X = 0$.

The Yule-Walker equations - Example

Compute the covariance function, $r_X(\tau)$, of the process X_t, $t = 0, \pm 1, \pm 2, \ldots$, given by

$$X_t - X_{t-1} + 0.5X_{t-2} = e_t,$$

where e_t, $t = 0, \pm 1, \pm 2 \ldots$, is zero-mean white noise with variance $\sigma^2 = 1$.

The Yule-Walker equations - Solution

The covariance function can be found from

$$C[X_{t-\tau}, X_t - X_{t-1} + 0.5X_{t-2}] = C[X_{t-\tau}, e_t]$$

for $\tau = 0, 1, 2, \ldots$. giving the Yule-Walker equations,

$$\begin{aligned}
r_X(0) - r_X(-1) + 0.5r_X(-2) &= 1, \quad \tau = 0, \\
r_X(1) - r_X(0) + 0.5r_X(-1) &= 0, \quad \tau = 1, \\
r_X(2) - r_X(1) + 0.5r_X(0) &= 0, \quad \tau = 2, \\
&\ldots \\
r_X(\tau) - r_X(\tau - 1) + 0.5r_X(\tau - 2) &= 0, \quad \tau \geq 2.
\end{aligned}$$

The Yule-Walker equations - general case

For an AR(p)-process,

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t,$$

where, e_t, $t = 0, \pm 1, \pm 2 \ldots$, is zero-mean white noise with variance σ^2, the covariance function $r_X(\tau)$, is computed from the Yule-Walker equations defined as

$$r_X(\tau) + a_1 r_X(\tau - 1) + \ldots + a_p r_X(\tau - p) = \begin{cases} \sigma^2 & \text{for } \tau = 0 \\ 0 & \text{for } \tau = 1, 2, \ldots \end{cases}$$

The Yule-Walker equations - Solution

With $r_X(-1) = r_X(1)$ and $r_X(-2) = r_X(2)$ we get the first 3 equations as

$$\begin{aligned} r_X(0) - r_X(1) + 0.5r_X(2) &= 1, \\ r_X(1) - r_X(0) + 0.5r_X(1) &= 0, \\ r_X(2) - r_X(1) + 0.5r_X(0) &= 0, \end{aligned}$$

giving $r_X(0) = 2.4$, $r_X(\pm 1) = 1.6$ and $r_X(\pm 2) = 0.4$. (The detailed solution is found in Example8.pdf, page 2, among the lecture notes.) Further values of the covariance function can then be found iteratively from

$$r_X(\tau) - r_X(\tau - 1) + 0.5r_X(\tau - 2) = 0, \quad \tau \geq 2,$$

such as $r_X(\pm 3) = -0.4$, $r_X(\pm 4) = -0.6$...

Estimation of AR-parameters

Assuming that we have a series of measurements, x_t, $t = 1 \ldots n$, we can also directly use the AR-structure and the Yule-Walker equations,

$$\hat{r}_X(\tau) + a_1 \hat{r}_X(\tau - 1) + \ldots + a_p \hat{r}_X(\tau - p) = \begin{cases} \hat{\sigma}^2 & \text{for } \tau = 0 \\ 0 & \text{for } \tau = 1, 2, \ldots \end{cases}$$

to estimate the AR-parameters $a_1 \ldots a_p$. See chapter 7.3 for a matrix solution. However, often the model order p is difficult to estimate, and errors lead to either spurious peaks or missing information.

The spectral density of an AR(p)-process

The AR(p)-process is given by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t.$$

We rewrite the process as

$$e_t = X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p},$$

where e_t is now the output from an LTI filter with the input X_t, i.e.

$$e_t = \sum_{k=0}^{p} a_k X_{t-k} = \sum_{k=0}^{p} h(k) X_{t-k}.$$

Then

$$R_e(f) = |H(f)|^2 R_X(f) = |\sum_{k=0}^{p} a_k e^{-i2\pi fk}|^2 R_X(f).$$

The spectral density of an AR(p)-process

With

$$R_e(f) = |H(f)|^2 R_X(f) = |\sum_{k=0}^{p} a_k e^{-i2\pi fk}|^2 R_X(f),$$

we reformulate back into

$$R_X(f) = \frac{R_e(f)}{|H(f)|^2} = \frac{R_e(f)}{|\sum_{k=0}^{p} a_k e^{-i2\pi fk}|^2},$$

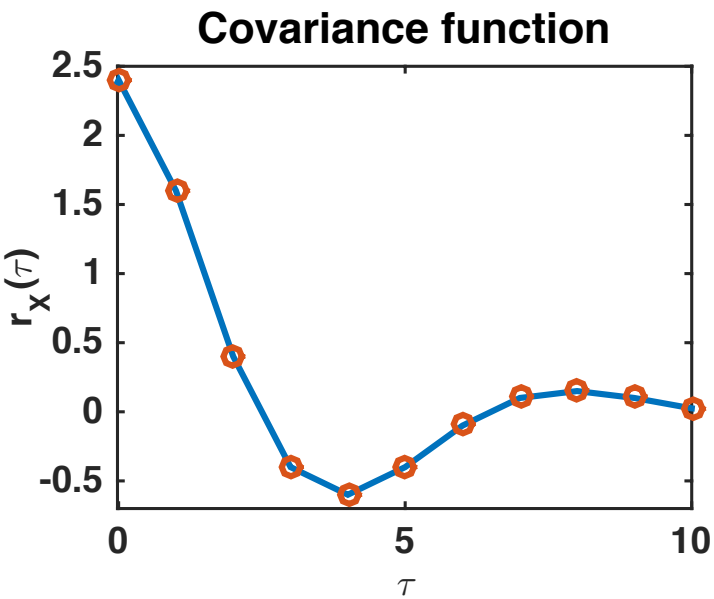
where $R_e(f) = \sigma^2$. The spectral density of the AR(p)-process is

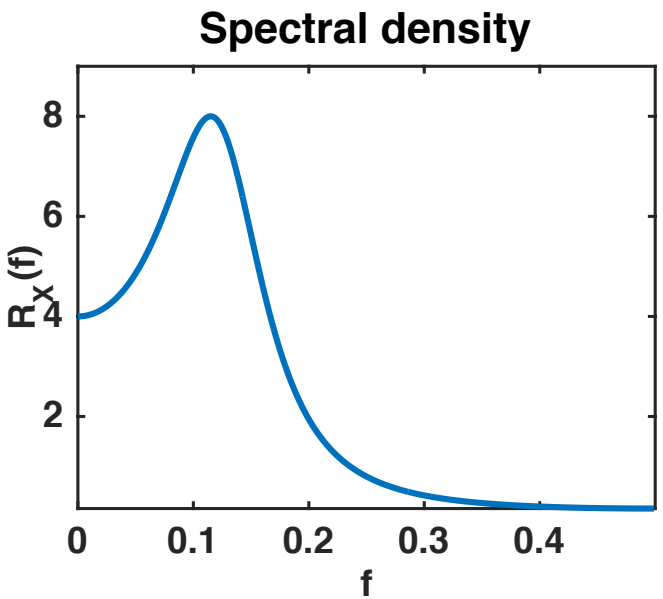
$$R_X(f) = \frac{\sigma^2}{|1 + a_1 e^{-i2\pi f} + \ldots + a_p e^{-i2\pi fp}|^2}, \quad -1/2 < f \leq 1/2.$$

The AR(2)-process example

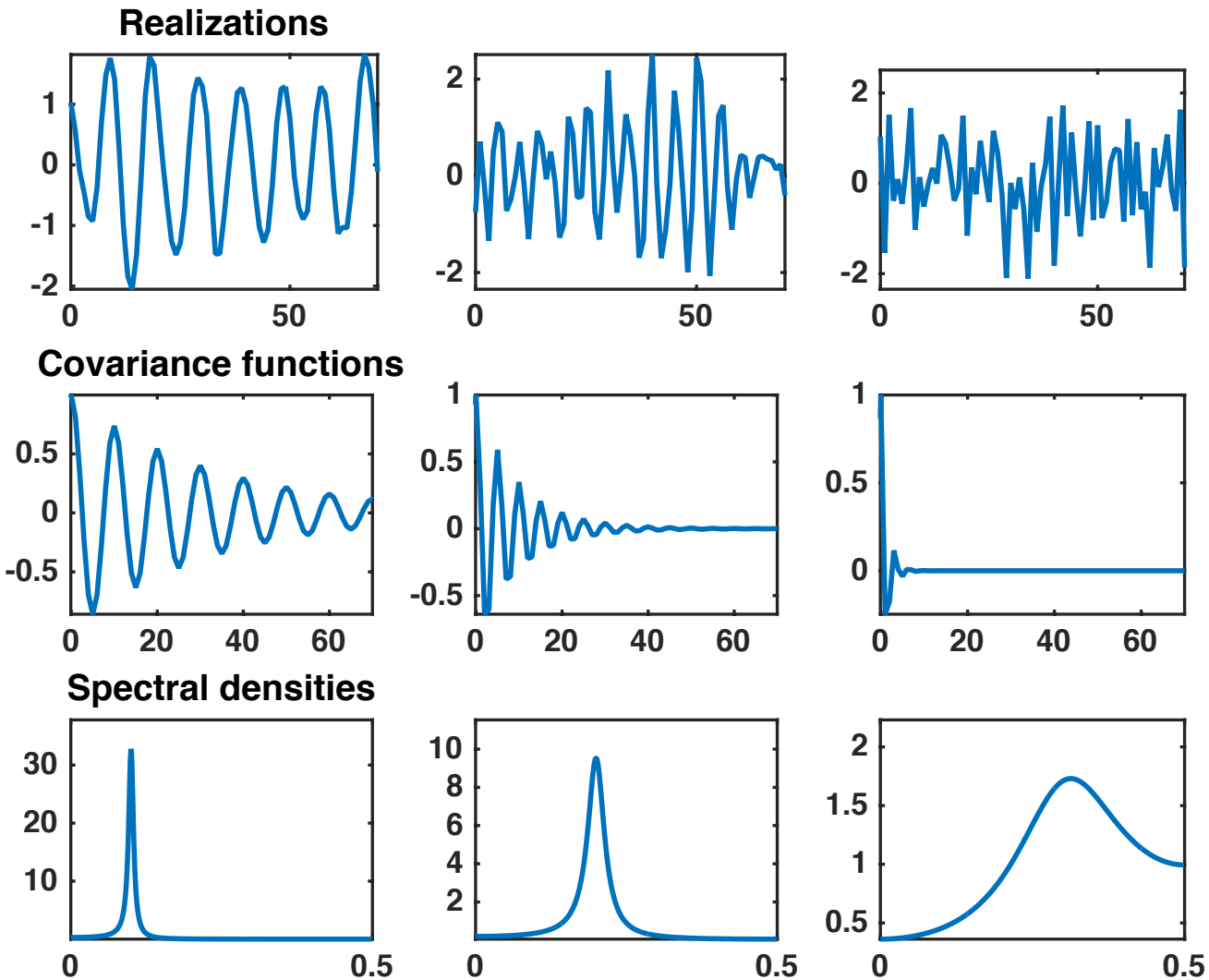
With the parameters $a_1 = -1$ and $a_2 = 0.5$, the spectral density is

$$R_X(f) = \frac{1}{|1 - e^{-i2\pi f} + 0.5e^{-i2\pi f 2}|^2}.$$





Examples of AR(2)-processes



Poles of an AR(2)-process

For the AR(2)-process we get

$$R_X(f) = |\frac{1}{1 + a_1 e^{-i2\pi f} + a_2 e^{-i4\pi f}}|^2 \sigma^2 =$$

$$= |\frac{e^{i4\pi f}}{e^{i4\pi f} + a_1 e^{i2\pi f} + a_2}|^2 \sigma^2,$$

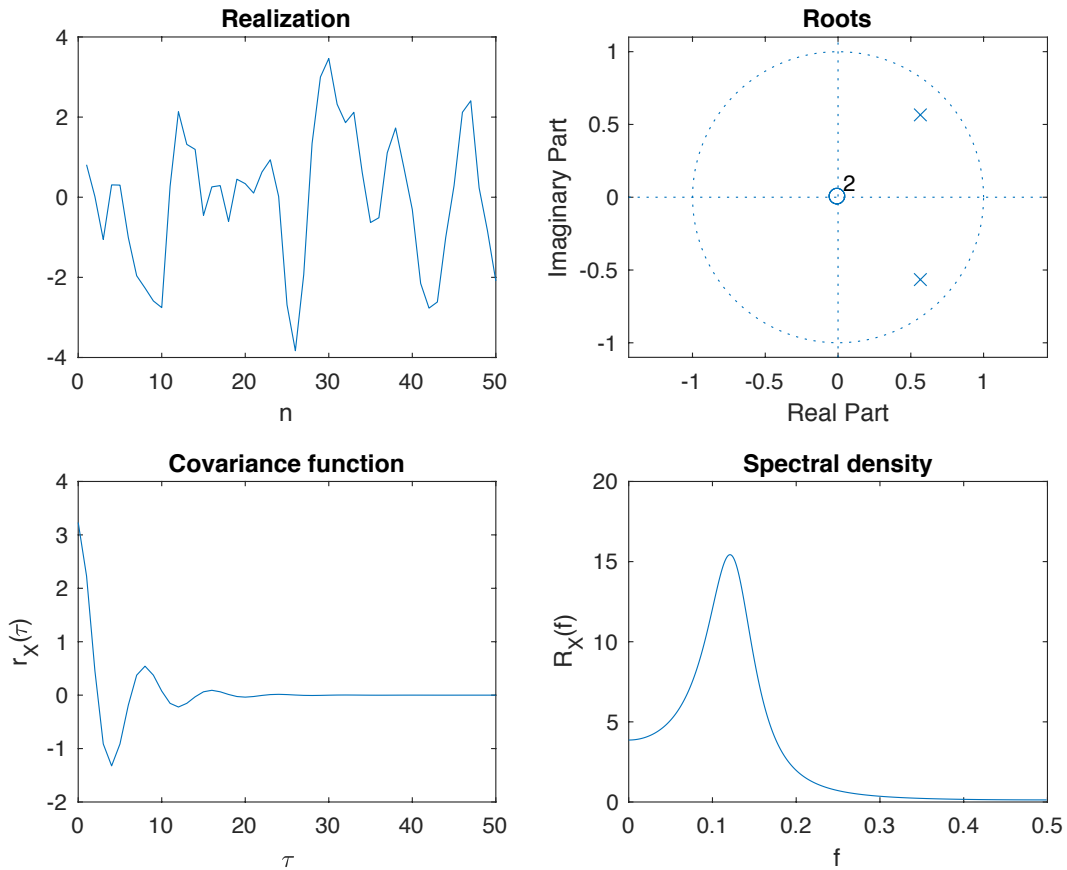
and we can then write the factorized spectral density as

$$R_X(f) = |\frac{e^{i4\pi f}}{(e^{i2\pi f} - z_1^P)(e^{i2\pi f} - z_2^P)}|^2 \sigma^2.$$

The roots $z_1^P = \rho_0 e^{i2\pi f_0}$ and $z_2^P = \rho_0 e^{-i2\pi f_0}$ are called **poles** (poler). For an AR(2)-process there are always 2 complex conjugated poles and 2 zeros located at the origin.

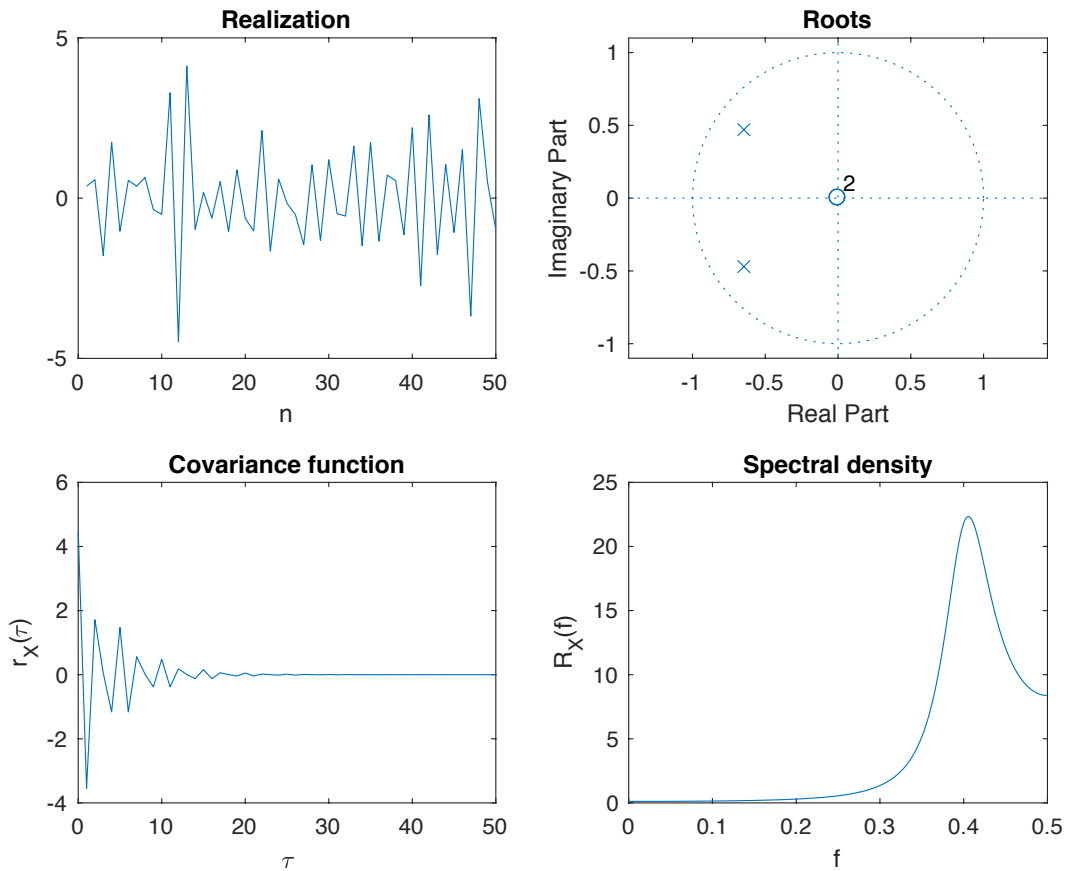
Example AR(2)-process

When the roots are located with angle $\omega_0 = 2\pi f_0$, (from the real axis), the peaks of the spectral density are at f_0.



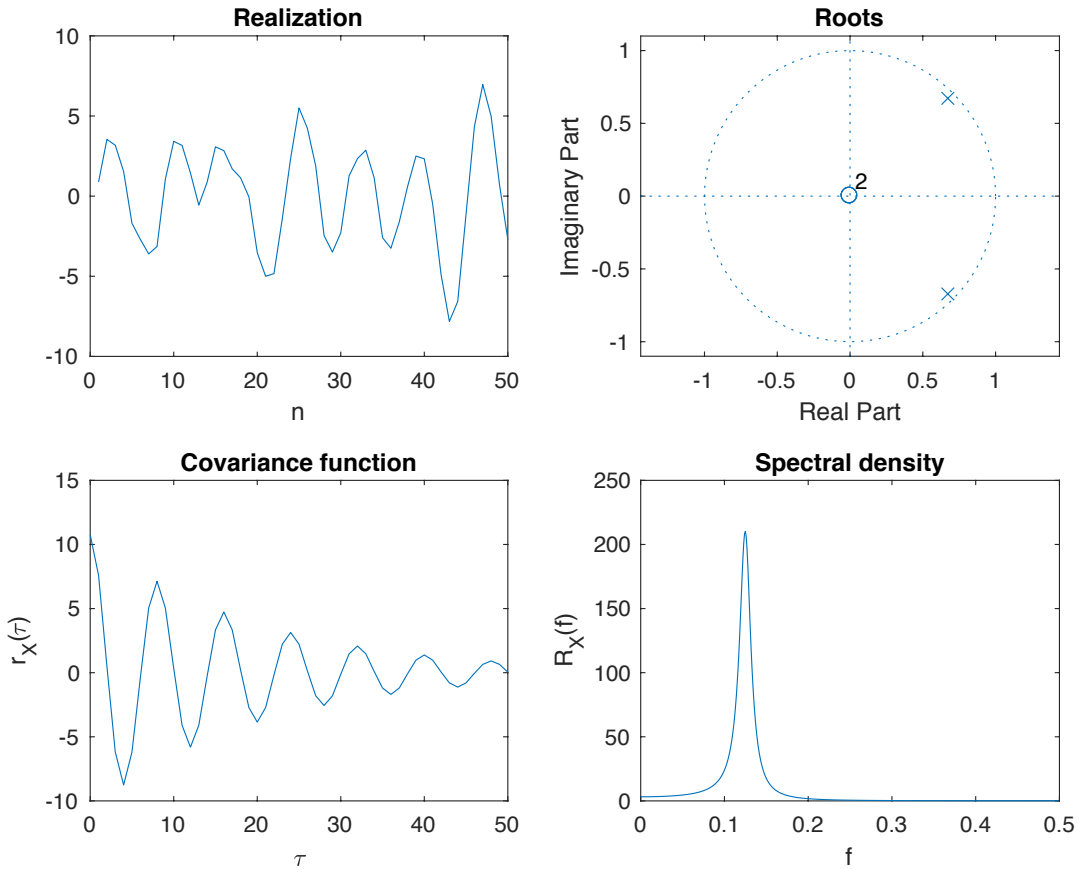
Example AR(2)-process

When the roots are located at a larger angle ω_0, the peak of the spectral density is at a higher frequency f_0, and covariance function will have shorter time of periods.



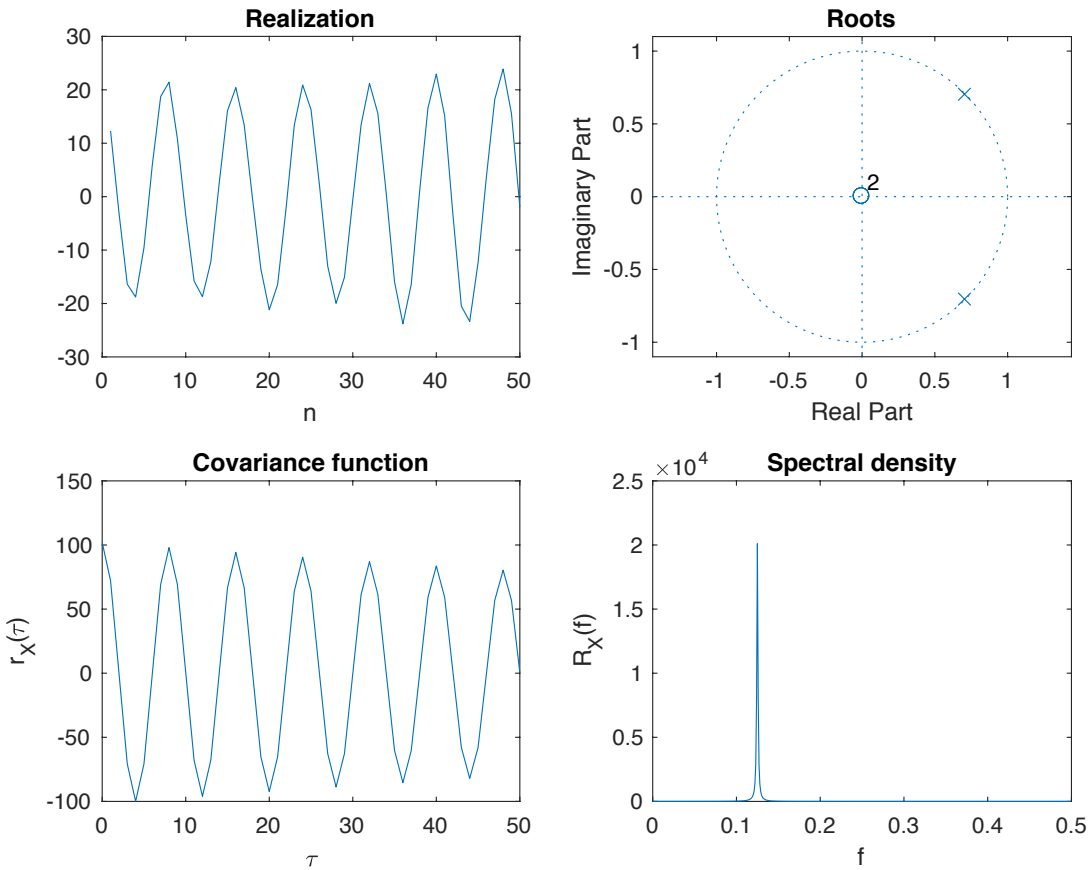
Example AR(2)-process

The closer to the unit circle the pole at angle ω_0 is, the sharper and higher the peak of the spectral density at f_0 will be. The covariance function will be less damped.



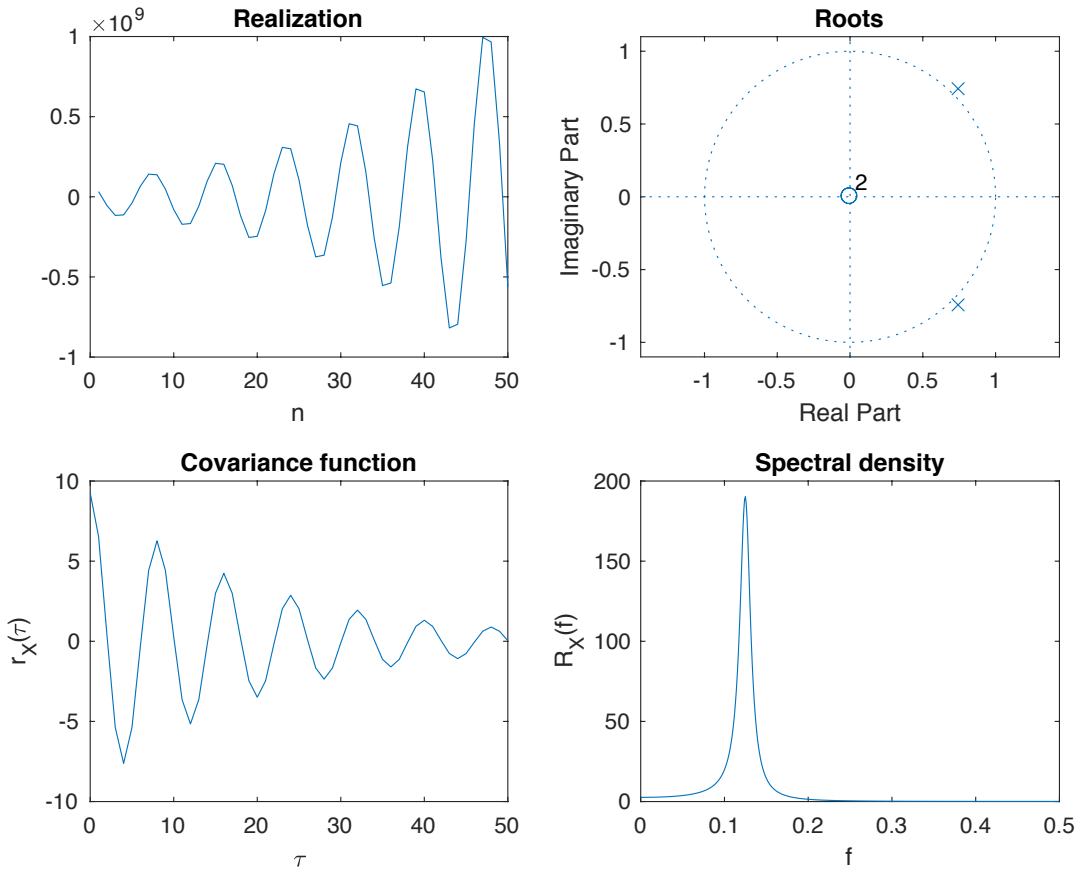
Example AR(2)-process

When the roots are located very close to the unit circle, the spectral density peaks are very high and sharp and the covariance function is dependent for high values of $|\tau|$.

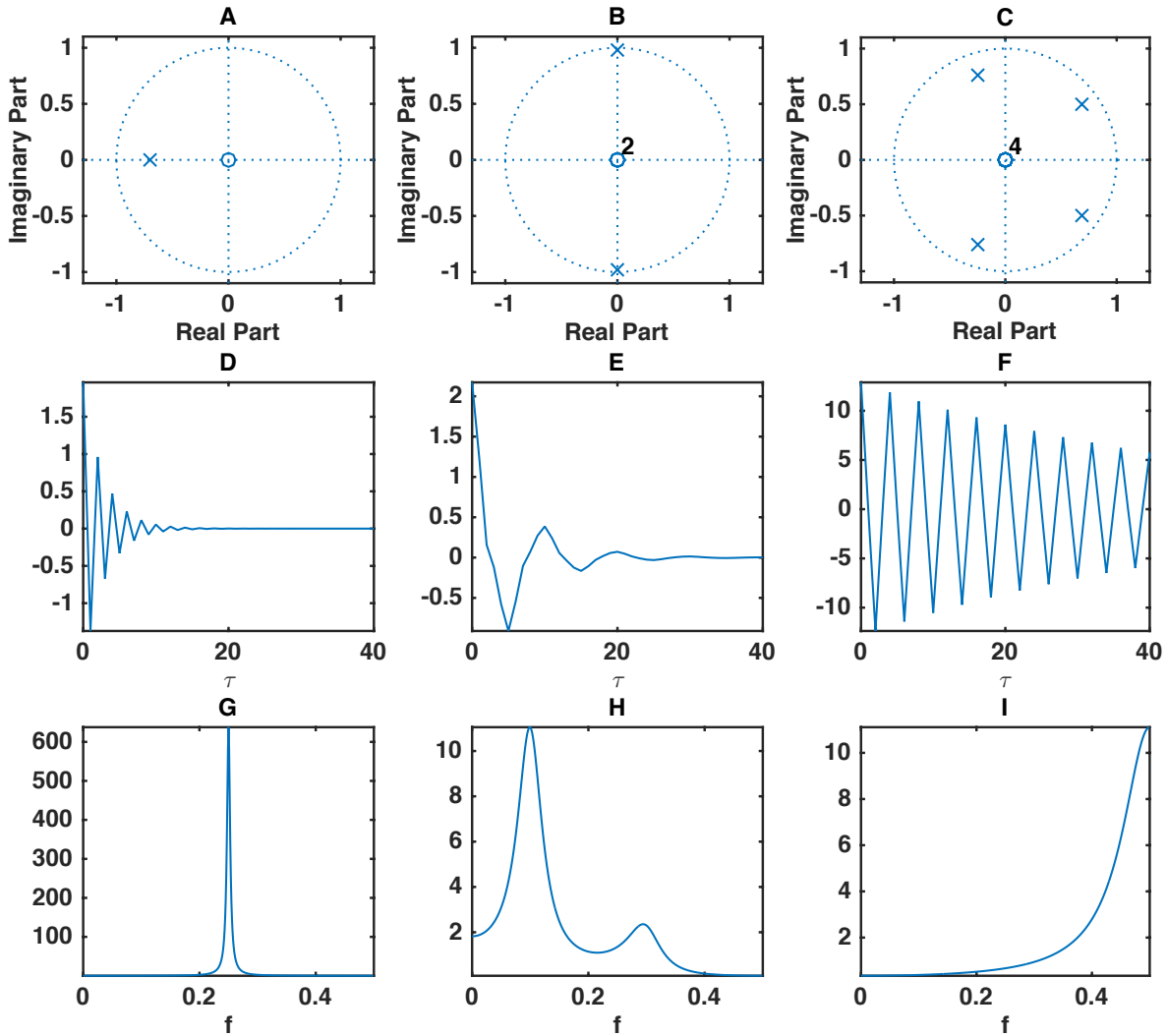


Example AR(2)-process

When the roots are located outside the unit circle, the process is not stable, although the covariance function and spectral density seem to be nice.



Example of AR(p)-processes



Zeros of an MA(2)-process

With $c_0 = 1$, the spectral density can also be written as

$$\begin{aligned} R_X(f) &= |H(f)|^2\sigma^2 = |1 + c_1 e^{-i2\pi f} + c_2 e^{-i4\pi f}|^2\sigma^2, \\ &= \left|\frac{e^{i4\pi f} + c_1 e^{i2\pi f} + c_2}{e^{i4\pi f}}\right|^2\sigma^2, \end{aligned}$$

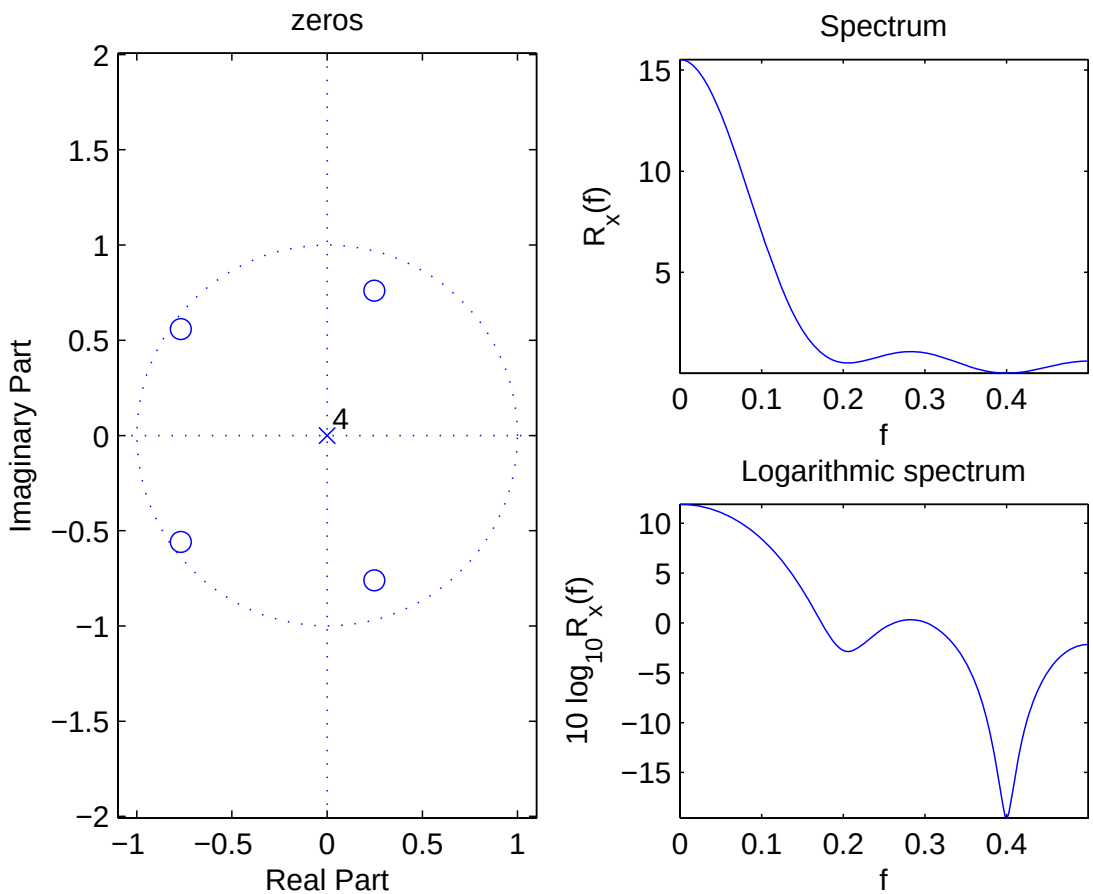
and we can then write the factorized spectral density as

$$R_X(f) = |(e^{i2\pi f} - z_1^C)(e^{i2\pi f} - z_2^C)|^2\sigma^2.$$

The roots $z_1^C = \rho_0 e^{i2\pi f_0}$ and $z_2^C = \rho_0 e^{-i2\pi f_0}$ are called **zeros**. For an MA(2)-process there are always 2 complex conjugated zeros and 2 poles located at the origin.

Example

The closer the zero is to the unit circle, the closer to zero is the spectral density at the specific frequency. Often the dB-scale, i.e., $10 \log_{10} R_X(f)$, is used to view the location of zeros.



The ARMA(p,q)-process

A combination of an MA-process and an AR-process gives a general model called an ARMA-model. An ARMA(2,2)-model is written

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} = e_t + c_1 e_{t-1} + c_2 e_{t-2}.$$

The spectral density is

$$R_x(f) = \sigma^2 \left| \frac{\sum_{k=0}^{2} c_k e^{-i2\pi fk}}{\sum_{k=0}^{2} a_k e^{-i2\pi fk}} \right|^2,$$

and the numerator can be decomposed into zeros and the denominator can be decomposed into poles.

Example ARMA(2,2)-process