Stationary stochastic processes

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Lecture 8

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Yesterday news: Filtering of stationary processes

From the continuous time stationary input process, $X(t)$, $t \in \mathbb{R}$ the stationary output process, $Y(t)$, $t \in \mathbb{R}$, is given as

$$Y(t) = \int_{-\infty}^{\infty} h(t-u)X(u)du = \int_{-\infty}^{\infty} h(u)X(t-u)du.$$

For a causal filter $h(t) = 0$, for $t < 0$. A stationary Gaussian process is filtered to a stationary Gaussian process.

$h(t) = \int H(f)e^{i2\pi ft}df$ $\qquad$ $H(f) = \int h(t)e^{-i2\pi ft}dt$

$m_Y = m_X \int h(u)du$ $\qquad$ $m_Y = m_X H(0)$

$r_Y(\tau) = \int\int h(u)h(v)r_X(\tau + u - v)du\,dv$ $\qquad$ $R_Y(f) = |H(f)|^2 R_X(f)$

Schedule for today

- Continuation chapter 6.2: Discrete time filters. Example
- Chapter 6.3.2: Differentiation - expected value and covariance of the derivative. Example
- Spectral density relations for the derivative. Example.
- Chapter 6.5: Cross-covariance
- Cross-covariance including derivatives. Example.
- Chapters 6.5 and 9.5.4: Cross-spectrum and coherence spectrum. Estimation.

Filtering of discrete time processes

Definition 6.1: The output process Y_t, $t = 0, \pm 1, \pm 2 \ldots$, is obtained from the input X_t, $t = 0, \pm 1, \pm 2 \ldots$, through

$$Y_t = \sum_{u=-\infty}^{\infty} h(t-u) X_u = \sum_{u=-\infty}^{\infty} h(u) X_{t-u},$$

where $h(t)$ is the impulse response.

Definition 6.2: The frequency function $H(f)$ of the discrete time impulse response is defined as

$$H(f) = \sum_{t=-\infty}^{\infty} h(t) e^{-i2\pi f t},$$

with the inverse transform as

$$h(t) = \int_{-1/2}^{1/2} H(f) e^{i2\pi f t} df.$$

Relations

Mean value:

$$m_Y = m_X \sum_{u=-\infty}^{\infty} h(u) = m_X H(0).$$

Covariance function:

$$r_Y(\tau) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h(u)h(v)r_X(\tau + u - v).$$

Spectral density:

$$R_Y(f) = |H(f)|^2 R_X(f).$$

Example

A discrete time filter is defined as

$$h(0) = 1/2, \quad h(2) = 1/2,$$

and zero for all other values. The input process, X_t, $t = 0, \pm 1, \pm 2 \ldots$ is a zero-mean stationary process with covariance function

$$r_X(0) = 1, \quad r_X(\pm 1) = -1/2.$$

Calculate the spectral density and the covariance function of the output process.

Solution 1

The spectral density of X_t is

$$R_X(f) = \sum_{\tau} r_X(\tau) e^{-i2\pi f \tau} = 1 - \cos(2\pi f).$$

The frequency function is

$$H(f) = \sum_{u} h(u) e^{-i2\pi f u} = \frac{1}{2} + \frac{1}{2} e^{-i2\pi f 2}.$$

The spectral density of the output process Y_t is

$$R_Y(f) = |H(f)|^2 R_X(f) = (\frac{1}{2} + \frac{1}{2}\cos(4\pi f)) \cdot (1 - \cos(2\pi f)) =$$

$$... = \frac{1}{2} - \frac{3}{4}\cos(2\pi f) + \frac{1}{2}\cos(4\pi f) - \frac{1}{4}\cos(6\pi f).$$

Solution 2

The covariance function for Y_t is

$$r_Y(\tau) = \sum_u \sum_v h(u)h(v)r_X(\tau + u - v) =$$

$$\frac{1}{4}(2r_X(\tau) + r_X(\tau + 2) + r_X(\tau - 2)),$$

with $r_Y(0) = \frac{1}{2}$, $r_Y(1) = -\frac{3}{8}$, $r_Y(2) = \frac{1}{4}$, $r_Y(3) = -\frac{1}{8}$ and $r_Y(\tau) = 0$ for $\tau > 3$. Symmetry gives $r_Y(-\tau) = r_Y(\tau)$. Check that $r_Y(0)$ is positive and has the largest value! The spectral density is given as

$$R_Y(f) = \sum_{\tau=-3}^{3} r_Y(\tau)e^{-i2\pi f\tau} =$$

$$= \frac{1}{2} - \frac{3}{4}\cos(2\pi f) + \frac{1}{2}\cos(4\pi f) - \frac{1}{4}\cos(6\pi f).$$

Differentiation

To solve a linear stochastic differential equation (SDE), with $Y(t)$ and $X(t)$, $t \in \mathbb{R}$, e.g.,

$$Y''(t) + a_1 Y'(t) + a_0 Y(t) = X(t),$$

we need some knowledge of derivatives and integrals of stochastic processes. (For the interested reader, see chapter 8.2.2)

Differentiation

A stochastic process $X(t)$, $t \in \mathbb{R}$, is said to be differentiable (deriverbar) in quadratic mean with the derivative $X'(t)$ if

$$\frac{X(t+h) - X(t)}{h} \to X'(t),$$

when $h \to 0$ for all $t \in \mathbb{R}$, i.e., if

$$E\left[\left(\frac{X(t+h) - X(t)}{h} - X'(t)\right)^2\right] \to 0,$$

when $h \to 0$.

Differentiation

The derivative $X'(t)$, $t \in \mathbb{R}$, of a weakly stationary process $X(t)$, $t \in \mathbb{R}$, is weakly stationary. The derivative $X'(t)$ of a Gaussian process $X(t)$ is also a Gaussian process.

The expected value of the derivative is

$$m_{X'} = 0,$$

as

$$\begin{aligned} m_{X'} &= E[\lim_{h\to 0} \frac{X(t+h) - X(t)}{h}] = \lim_{h\to 0} \frac{E[X(t+h)] - E[X(t)]}{h} \\ &= \lim_{h\to 0} \frac{m_X - m_X}{h} = 0. \end{aligned}$$

Differentiation

The covariance function of the derivative relates to the process covariance as

$$r_{X'}(\tau) = -r''_X(\tau).$$

Proof:

$$\begin{aligned} r_{X'}(\tau) &= C\left[\lim_{k\to 0}\frac{X(t+k)-X(t)}{k}, \lim_{h\to 0}\frac{X(t+\tau+h)-X(t+\tau)}{h}\right] \\ &= \lim_{h\to 0}\frac{1}{h}\lim_{k\to 0}\left(\frac{r_X(\tau+h-k)-r_X(\tau+h)}{k}-\frac{r_X(\tau-k)-r_X(\tau)}{k}\right) \\ &= \lim_{h\to 0}\frac{-r'_X(\tau+h)+r'_X(\tau)}{h} = -r''_X(\tau). \end{aligned}$$

Example

A Gaussian stationary process $X(t)$, $t \in \mathbb{R}$, has covariance function

$$r_X(\tau) = \exp(-\tau^2/2).$$

Calculate

$$P(X'(t) > 1).$$

Solution

The mean value of the differentiated process $X'(t)$ is always zero, i.e.,

$$E[X'(t)] = 0.$$

The variance is

$$V[X'(t)] = r_{X'}(0) = -r''_X(0) = e^{-\tau^2/2} - \tau^2 e^{-\tau^2/2}|_{\tau=0} = 1.$$

Then follows

$$P(X'(t) > 1) = 1 - \Phi(1) = 0.159.$$

Differentiation, spectral density

To find the spectral density of $X'(t)$ we differentiate the definition $r_X(\tau) = \int e^{i2\pi f\tau}\, R_X(f)\, df$ twice.
We get

$$r_{X'}(\tau) = -r_X''(\tau) = \int e^{i2\pi f\tau}\,(2\pi f)^2\, R_X(f)\, df = \int e^{i2\pi f\tau}\, R_{X'}(f)\, df,$$

i.e. the spectral density of $X'(t)$ is identified

$$R_{X'}(f) = (2\pi f)^2 R_X(f).$$

The spectral density $R_{X'}(f)$ is positive and symmetric as $R_X(f)$ is positive and symmetric. It is also integrable if

$$\int_{-\infty}^{\infty} R_{X'}(f)df = \int_{-\infty}^{\infty} (2\pi f)^2 R_X(f)df < \infty.$$

Differentiation, summary

Let $X(t)$, $t \in \mathbb{R}$, be a weakly stationary process with covariance function $r_X(\tau)$. Then the following statements are equivalent:

- $X(t)$ is differentiable in quadratic mean
- $r_X(\tau)$ is two times differentiable for every τ
- $\int_{-\infty}^{\infty} (2\pi f)^2 R_X(f) df < \infty$

Example

Let $X(t)$, $t \in \mathbb{R}$, be a stationary Gaussian process with the spectral density,

$$R_X(f) = \begin{cases} 1 & \text{for } |f| \leq 1, \\ 0 & \text{for all other values.} \end{cases}$$

Show that the process,

$$Y(t) = \int_0^\infty e^{-u} X(t-u) du,$$

is differentiable in quadratic mean and determine $V[Y'(t)]$.

Solution

The impulse response of the filter is $h(t) = e^{-t}$, $t > 0$, and frequency function, $H(f) = 1/(1 + i2\pi f)$. We find

$$R_Y(f) = |H(f)|^2 R_X(f) = \frac{1}{|1 + i2\pi f|^2} = \frac{1}{1 + (2\pi f)^2} \text{ for } |f| \leq 1,$$

and

$$\int_{-\infty}^{\infty} (2\pi f)^2 R_Y(f) df = \int_{-1}^{1} (2\pi f)^2 \frac{1}{1 + (2\pi f)^2} df < \infty.$$

Accordingly, $Y(t)$ is differentiable in quadratic mean. The variance of the derivative is

$$V[Y'(t)] = \int_{-\infty}^{\infty} (2\pi f)^2 R_Y(f) df = \int_{-1}^{1} \frac{(2\pi f)^2}{1 + (2\pi f)^2} df = 2 - \frac{1}{\pi} \arctan(2\pi).$$

Back to correlation

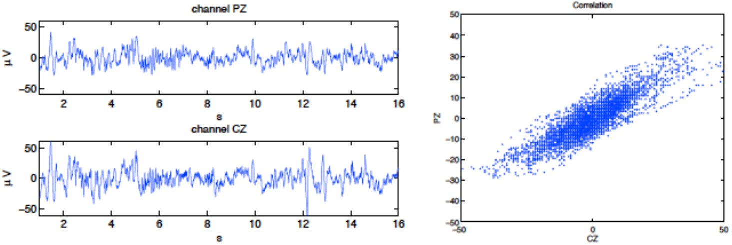


Figure 2.5 *Observed EEG-signals from two different channels (left) scatter plot of* x_t, y_t*. The estimated correlation coefficient between* X_t *and* Y_t *is* 0.9.

Cross-covariance

Definition 6.5: The cross-covariance is defined as

$$r_{X,Y}(\tau) = C[X(t), Y(t+\tau)] = E[X(t)Y(t+\tau)] - m_X m_Y.$$

It is easily shown that

$$r_{Y,X}(\tau) = r_{X,Y}(-\tau),$$

but in general the cross-covariance is not symmetric, i.e.

$$r_{X,Y}(\tau) \neq r_{X,Y}(-\tau),$$

Cross-covariance of derivatives

If $X(t)$, $t \in \mathbb{R}$, is differentiable in quadratic mean, then $X(t)$ and $X'(t)$ have a cross-covariance function,

$$
\begin{aligned}
r_{X,X'}(\tau) &= C[X(t), X'(t+\tau)] = C[X(t), \frac{X(t+\tau+h)-X(t+\tau)}{h}], \\
&= \frac{1}{h}(C[X(t), X(t+\tau+h)] - C[X(t), X(t+\tau)]), \\
&= \frac{1}{h}(r_X(\tau+h) - r_X(\tau)) = r'_X(\tau),
\end{aligned}
$$

when $h \to 0$.

We know that $r_X(\tau) = r_X(-\tau)$ is a symmetric even function so $r'_X(\tau) = -r'_X(-\tau)$ is odd and therefore

$$r_{X,X'}(0) = r'_X(0) = 0.$$

(Theorem 6.4: p. 149)

Cross-covariance of derivatives

Note that the cross-covariance function of $X'(t)$ and $X(t)$, i.e. the shifted order, is

$$r_{X',X}(\tau) = r_{X,X'}(-\tau) = r'_X(-\tau) = -r'_X(\tau).$$

A general formulation of the cross-covariance of derivatives

$$r_{X^{(j)},X^{(k)}}(\tau) = (-1)^j r_X^{(j+k)}(\tau),$$

included in the table of formulas.

Example

The continuous time stationary Gaussian process $X(t)$, $t \in \mathbb{R}$, has expected value $E[X(t)] = 2$ and covariance function

$$r_X(\tau) = \exp(-\tau^2/2).$$

A new process $Y(t)$, $t \in \mathbb{R}$, is created as

$$Y(t) = 5X(t) - 2X'(t-1).$$

Calculate the the expected value, $E[Y(t)]$ and the variance, $V[Y(t)]$.

Solution

The expected value is

$$E[Y(t)] = 5E[X(t)] - 2E[X'(t-1)] = 10$$

The variance is

$$V[Y(t)] = V[5X(t) - 2X'(t-1)] =$$

$$25r_X(0) + 4r_{X'}(0) - 10r_{X,X'}(-1) - 10r_{X',X}(1) =$$

$$25r_X(0) + 4r_{X'}(0) - 20r_{X,X'}(-1) =$$

$$25r_X(0) + 4(-r_X''(0)) - 20r_X'(-1).$$

The covariances and cross-covariances are given by $r_X(0) = 1$, $r_X'(-1) = e^{-1/2}$, $r_X''(0) = -1$, and

$$V[Y(t)] = 25 + 4 - 20e^{-1/2} \approx 16.87.$$

Cross-spectrum

The cross-spectrum, $R_{X,Y}(f)$, in continuous time is defined as

$$R_{X,Y}(f) = \int_{-\infty}^{\infty} r_{X,Y}(\tau) e^{-i2\pi f \tau} d\tau,$$

and

$$r_{X,Y}(\tau) = \int_{-\infty}^{\infty} R_{X,Y}(f) e^{i2\pi f \tau} df.$$

The corresponding formulas in discrete time are:

$$R_{X,Y}(f) = \sum_{\tau=-\infty}^{\infty} r_{X,Y}(\tau) e^{-i2\pi f \tau},$$

and

$$r_{X,Y}(\tau) = \int_{-1/2}^{1/2} R_{X,Y}(f) e^{i2\pi f \tau} df.$$

(Theorem 6.6)

Cross-amplitude and phase spectrum

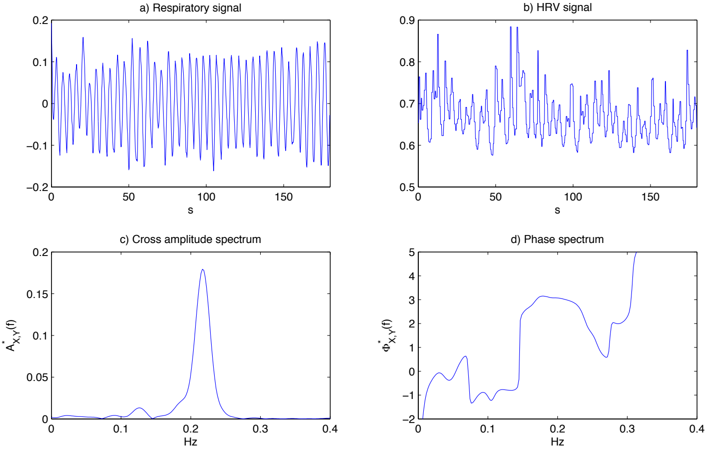
The cross-spectrum is complex valued,

$$R_{X,Y}(f) = A_{X,Y}(f)e^{i\Phi_{X,Y}(f)},$$

where $A_{X,Y}(f) = |R_{X,Y}(f)|$ is the cross-amplitude spectrum and $\Phi_{X,Y}(f) = \arg R_{X,Y}(f)$ is the phase spectrum.

Example

The respiratory signal and the heart rate variability signal can be analysed using the cross-spectrum.



Coherence spectrum

The (squared) coherence spectrum is defined as

$$\kappa^2_{X,Y}(f) = \frac{|R_{X,Y}(f)|^2}{R_X(f)R_Y(f)},$$

and $0 \leq \kappa^2_{X,Y} \leq 1$.

Estimation of coherence spectrum

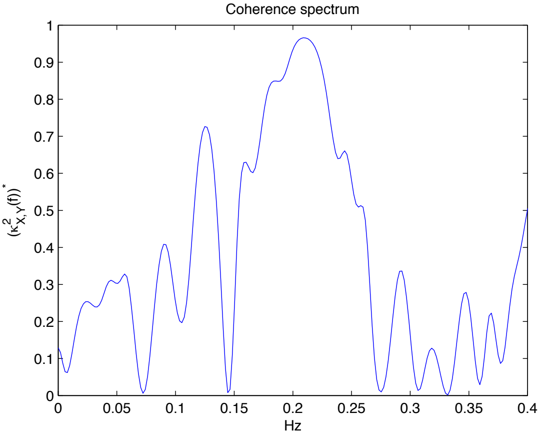
If we use the periodogram for estimation of the coherence spectrum

$$\widehat{\kappa_{x,y}^2} = \frac{|\widehat{R}_{x,y}(f)|^2}{\widehat{R}_x(f)\widehat{R}_y(f)} = \frac{\frac{1}{n}\mathcal{X}(f)\mathcal{Y}(f)^* \cdot \frac{1}{n}\mathcal{X}(f)^*\mathcal{Y}(f)}{\frac{1}{n}\mathcal{X}(f)\mathcal{X}(f)^* \cdot \frac{1}{n}\mathcal{Y}(f)\mathcal{Y}(f)^*} = ?,$$

where $\mathcal{X}(f)$ and $\mathcal{Y}(f)$ are the discrete Fourier transforms of the two sequences, the resulting estimate is obviously worthless. The solution is to use the Welch method or some multitaper approach!

Example

We then find that the respiratory signal and the heart rate variability signal have strong correlation around 0.2 Hz.



Filtering with disturbance

A common model is

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du + Z(t),$$

where $X(t)$, $t \in \mathbb{R}$, is the input process and $Z(t)$, $t \in \mathbb{R}$, the disturbance process. The processes $X(t)$ and $Z(t)$ are stationary and uncorrelated. We find the cross-covariance,

$$r_{X,Y}(\tau) = \int_{-\infty}^{\infty} h(u)r_X(\tau-u)du,$$

and the cross-spectral density

$$R_{X,Y}(f) = H(f)R_X(f).$$

For derivation of these expressions see Example 7-5.pdf among the lecture notes.

Estimation of frequency function

We then find

$$H(f) = \frac{R_{X,Y}(f)}{R_X(f)},$$

and an estimate of the frequency function is found as

$$\widehat{H}(f) = \frac{\widehat{R}_{x,y}(f)}{\widehat{R}_x(f)},$$

where we should use low variance estimators (the Welch method or some multitaper approach) for $\widehat{R}_{x,y}(f)$ and $\widehat{R}_x(f)$ as two estimates are divided.