Stationary stochastic processes

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Lecture 8

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Summary: Linear and time-invariant filters

From the continuous time input process, $X(t)$, $t \in T$ the output process, $Y(t)$, $t \in T$, is given as

$$Y(t) = \int_{-\infty}^{\infty} h(t-u)X(u)du = \int_{-\infty}^{\infty} h(u)X(t-u)du.$$

For a causal filter $h(t) = 0$, for $t < 0$. A stationary input process is filtered into a stationary output process. A stationary Gaussian process is filtered to a stationary Gaussian process.

$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft}df$	$H(f) = \int h(t)e^{-i2\pi ft}dt$
$m_Y = m_X \int h(u)du$	$m_Y = m_X H(0)$
$r_Y(\tau) = \int\int h(u)h(v)r_X(\tau + u - v)du\,dv$	$R_Y(f) = \|H(f)\|^2 R_X(f)$

Summary: Linear and time-invariant filters

From the discrete time input process, $X_t,\ t = 0, \pm1, \pm2, \ldots$, the output process, $Y_t,\ t = 0, \pm1, \pm2, \ldots$, is given as

$$Y_t = \sum_{u=-\infty}^{\infty} h(t-u)X_u = \sum_{u=-\infty}^{\infty} h(u)X_{t-u}.$$

For a causal filter $h(t) = 0$, for $t = -1, -2, -3, \ldots$. A stationary input process is filtered into a stationary output process. A stationary Gaussian process is filtered to a stationary Gaussian process.

$h(t) = \int_{-1/2}^{1/2} H(f)e^{i2\pi ft}df$	$H(f) = \sum h(u)e^{-i2\pi fu}$
$m_Y = m_X \sum h(u)$	$m_Y = m_X H(0)$
$r_Y(\tau) = \sum\sum h(u)h(v)r_X(\tau + u - v)$	$R_Y(f) = \lvert H(f)\rvert^2 R_X(f)$

Schedule for today

- Chapter 6.4: Continuous time white noise
- Chapter 6.3.2: Differentiation
- Chapters 6.5 and 9.5.4: Cross-spectrum

Continuous time white noise

A formal definition of white noise is made with

$$R(f) = R_0,$$

and

$$r(\tau) = R_0\delta(\tau).$$

The formal white noise is the limit of 'almost white noise', the Ornstein-Uhlenbeck process with a large α, se ex. 6.8, and it is also the derivative of the Wiener process (Brownian motion), often used in stochastic modeling.

Stochastic modeling

To solve a linear stochastic differential equation (SDE), where $X(t)$, $t \in T$, is a formal white noise process, e.g.,

$$Y''(t) + a_1 Y'(t) + a_0 Y(t) = X(t),$$

we need some knowledge of derivatives and integrals of stochastic processes. (For the interested reader, see chapter 8.2.2)

Differentiation

A stochastic process $X(t)$, $t \in T$, is said to be differentiable (deriverbar) in quadratic mean with the derivative $X'(t)$ if

$$\frac{X(t+h) - X(t)}{h} \to X'(t),$$

when $h \to 0$ for all $t \in T$, i.e., if

$$E\left[\left(\frac{X(t+h) - X(t)}{h} - X'(t)\right)^2\right] \to 0,$$

when $h \to 0$.

Differentiation

The derivative $X'(t)$, $t \in T$, of a weakly stationary process $X(t)$, $t \in T$, is weakly stationary with expected value,

$$m_{X'} = 0,$$

and covariance function,

$$r_{X'}(\tau) = -r''_X(\tau).$$

The derivative $X'(t)$ of a Gaussian process $X(t)$ is also a Gaussian process.

Differentiation

$$\begin{aligned} m_{X'} &= E[\lim_{h\to 0} \frac{(X(t+h)-X(t))}{h}] = \lim_{h\to 0} E[\frac{(X(t+h)-X(t))}{h}] \\ &= \lim_{h\to 0} \frac{(m_X - m_X)}{h} = 0. \end{aligned}$$

$$\begin{aligned} r_{X'}(\tau) &= C\left[\lim_{k\to 0} \frac{X(t+k)-X(t)}{k}, \lim_{h\to 0} \frac{X(t+\tau+h)-X(t+\tau)}{h}\right] \\ &= \lim_{h\to 0} \frac{1}{h} \lim_{k\to 0} \left(\frac{r_X(\tau+h-k)-r_X(\tau+h)}{k} - \frac{r_X(\tau-k)-r_X(\tau)}{k} \right) \\ &= \lim_{h\to 0} \frac{-r_X'(\tau+h)+r_X'(\tau)}{h} = -r_X''(\tau). \end{aligned}$$

Example

A Gaussian stationary process $X(t)$, $t \in T$, has covariance function

$$r_X(\tau) = \exp(-\tau^2/2).$$

Calculate

$$P(X'(t) > 1).$$

Example

The mean value of the differentiated process $X'(t)$ is always zero, i.e., $E[X'(t)] = 0$.
The variance is

$$V[X'(t)] = r_{X'}(0) = -r''_X(0) = e^{-\tau^2/2} - \tau^2 e^{-\tau^2/2}|_{\tau=0} = 1.$$

Then follows

$$P(X'(t) > 1) = 1 - \Phi(1) = 0.159.$$

Differentiation, spectral density

Differentiate the spectral representation $r_X(\tau) = \int e^{i2\pi f\tau} R_X(f)\, df$ twice. We get

$$r_X''(\tau) = -\int e^{i2\pi f\tau} (2\pi f)^2 R_X(f)\, df,$$

i.e. the spectral density of $X'(t)$ is

$$R_{X'}(f) = (2\pi f)^2 R_X(f).$$

The spectral density $R_{X'}(f)$ is positive and symmetric as $R_X(f)$ is positive and symmetric. It is also integrable if

$$\int_{-\infty}^{\infty} R_{X'}(f) df = \int_{-\infty}^{\infty} (2\pi f)^2 R_X(f) df < \infty.$$

Differentiation, summary

Let $X(t)$, $t \in T$, be a weakly stationary process with covariance function $r_X(\tau)$. Then the following statements are equivalent:

- $X(t)$ is differentiable in quadratic mean
- $r_X(\tau)$ is two times differentiable for every τ
- $\int_{-\infty}^{\infty} (2\pi f)^2 R_X(f) df < \infty$

Example

Let $X(t)$, $t \in T$, be a stationary Gaussian process with the spectral density,

$$R_X(f) = \begin{cases} 1 & \text{for } |f| \leq 1, \\ 0 & \text{for all other values.} \end{cases}$$

Show that the process,

$$Y(t) = \int_0^\infty e^{-u} X(t-u) du,$$

is differentiable in quadratic mean and determine $V[Y'(t)]$.

Solution

The impulse response of the filter is $h(t) = e^{-t}$, $t > 0$, and frequency function, $H(f) = 1/(1 + i2\pi f)$. We find

$$R_Y(f) = |H(f)|^2 R_X(f) = \frac{1}{|1 + i2\pi f|^2} = \frac{1}{1 + (2\pi f)^2} \text{ for } |f| \le 1,$$

and

$$\int_{-\infty}^{\infty} (2\pi f)^2 R_Y(f) df = \int_{-1}^{1} (2\pi f)^2 \frac{1}{1 + (2\pi f)^2} df < \infty.$$

Accordingly, $Y(t)$ is differentiable in quadratic mean. The variance of the derivative is

$$V[Y'(t)] = \int_{-\infty}^{\infty} (2\pi f)^2 R_Y(f) df = \int_{-1}^{1} \frac{(2\pi f)^2}{1 + (2\pi f)^2} df = 2 - \frac{1}{\pi} \arctan(2\pi).$$

Back to correlation

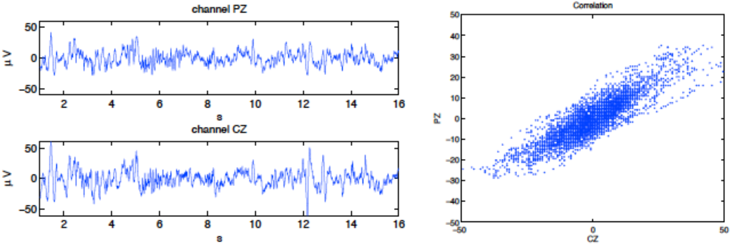


Figure 2.5 *Observed EEG-signals from two different channels (left) scatter plot of* x_t, y_t. *The estimated correlation coefficient between* X_t *and* Y_t *is* 0.9.

Cross-covariance

Definition 6.5: The cross-covariance is defined as

$$r_{X,Y}(\tau) = C[X(t), Y(t+\tau)] = E[X(t)Y(t+\tau)] - m_X m_Y.$$

The cross-covariance is not symmetric,

$$r_{X,Y}(\tau) \neq r_{X,Y}(-\tau),$$

but

$$r_{X,Y}(\tau) = r_{Y,X}(-\tau).$$

Cross-covariance of derivatives

If $X(t)$ is differentiable in quadratic mean, then $X(t)$ and $X'(t)$ have a cross-covariance function,

$$r_{X,X'}(\tau) = C[X(t), X'(t+\tau)] = r_X'(\tau),$$

and

$$r_{X',X}(-\tau) = r_X'(\tau).$$

We also find

$$r_{X,X'}(-\tau) = r_X'(-\tau) = -r_X'(\tau).$$

and

$$r_{X,X'}(0) = C[X(t), X'(t)] = r_X'(0) = 0.$$

Example

The continuous time stationary Gaussian process $X(t)$, $t \in T$, has expected value $E[X(t)] = 2$ and covariance function

$$r_X(\tau) = \exp(-\tau^2/2).$$

A new process $Y(t)$ is created as

$$Y(t) = 5X(t) - 2X'(t-1).$$

Calculate

$$P(Y(t) > 12).$$

Solution

The expected value is

$$E[Y(t)] = 5E[X(t)] - 2E[X'(t-1)] = 10$$

The variance is

$$V[Y(t)] = V[5X(t) - 2X'(t-1)] =$$

$$25r_X(0) + 4r_{X'}(0) - 10r_{X,X'}(-1) - 10r_{X',X}(1) =$$

$$25r_X(0) + 4(-r_X''(0)) - 20r_X'(-1).$$

The covariances and cross-covariances are given by $r_X(0) = 1$, $r_X'(-1) = e^{-1/2}$, $r_X''(0) = -1$, and

$$V[Y(t)] = 25 + 4 - 20e^{-1/2} \approx 16.87.$$

We get $P(Y(t) > 12) = 1 - \Phi(\frac{12-10}{\sqrt{16.87}}) = 1 - \Phi(0.49) \approx 0.31$.

Cross-spectrum

The cross-spectrum $R_{X,Y}(f)$, is defined as

$$R_{X,Y}(f) = \int_{-\infty}^{\infty} r_{X,Y}(\tau) e^{-i2\pi f\tau} d\tau,$$

and

$$r_{X,Y}(\tau) = \int_{-\infty}^{\infty} R_{X,Y}(f) e^{i2\pi f\tau} df.$$

The corresponding formulas in discrete time are:

$$R_{X,Y}(f) = \sum_{\tau=-\infty}^{\infty} r_{X,Y}(\tau) e^{-i2\pi f\tau},$$

and

$$r_{X,Y}(\tau) = \int_{-1/2}^{1/2} R_{X,Y}(f) e^{i2\pi f\tau} df.$$

(Theorem 6.6)

Cross-amplitude and phase spectrum

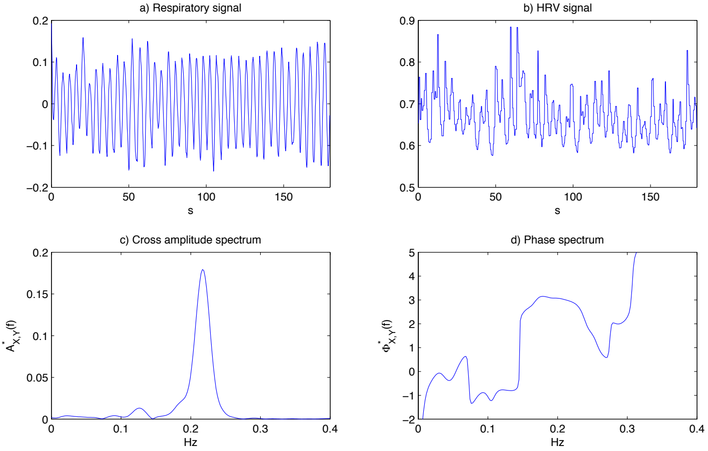
The cross-spectrum is complex valued,

$$R_{X,Y}(f) = A_{X,Y}(f)e^{i\Phi_{X,Y}(f)},$$

where $A_{X,Y}(f) = |R_{X,Y}(f)|$ is the cross-amplitude spectrum and $\Phi_{X,Y}(f) = \arg R_{X,Y}(f)$ is the phase spectrum.

Example

The respiratory signal and the heart rate variability signal can be analysed using the cross-spectrum.



Filtering with disturbance

A common model is

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du + Z(t),$$

where $X(t)$ is the input process and $Z(t)$ the disturbance process. The processes $X(t), Z(t)$, $t \in \mathbb{R}$, are stationary and uncorrelated.

Find the expressions for the cross-covariance, $r_{X,Y}(\tau)$, the cross-spectrum, $R_{X,Y}(f)$ and the frequency function $H(f)$.

Filtering with disturbance

We find the cross-covariance,

$$r_{X,Y}(\tau) = \int_{-\infty}^{\infty} h(u) r_X(\tau - u) du,$$

and the cross-spectrum,

$$R_{X,Y}(f) = H(f) R_X(f).$$

An expression for the frequency function is

$$H(f) = \frac{R_{X,Y}(f)}{R_X(f)}.$$

Estimation of frequency function

We can use the periodogram for estimation of the frequency function

$$\widehat{H}(f) = \frac{\widehat{R}_{x,y}(f)}{\widehat{R}_x(f)}.$$

The cross-spectrum estimate is

$$\widehat{R}_{x,y}(f) = \frac{1}{n}\mathcal{X}(f)\mathcal{Y}(f)^*,$$

where $\mathcal{X}(f) = \sum_{t=0}^{n-1} x_t e^{-i2\pi ft}$ and $\mathcal{Y}(f) = \sum_{t=0}^{n-1} y_t e^{-i2\pi ft}$. The variance of the estimate will be unreliable as two high variance estimates are divided. A better choice is the Welch method or some multitaper approach.

Coherence spectrum

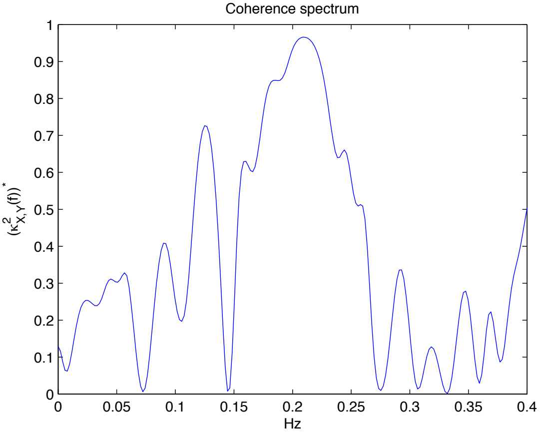
The (squared) coherence spectrum is defined as

$$\kappa^2_{X,Y}(f) = \frac{|R_{X,Y}(f)|^2}{R_X(f)R_Y(f)},$$

and $0 \leq \kappa^2_{X,Y} \leq 1$.

Example

The respiratory signal and the heart rate variability signal have strong correlation around 0.2 Hz.



Coherence for filtering and disturbance

For the filtered process with uncorrelated disturbance,

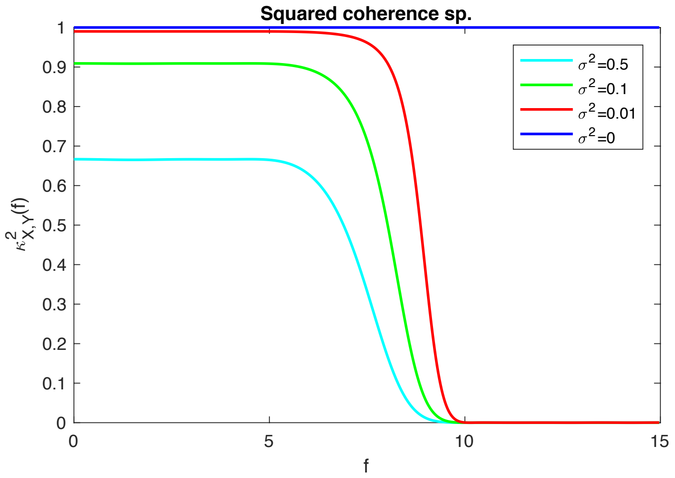
$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du + Z(t),$$

we find the output spectral density as $R_Y(f) = |H(f)|^2 R_X(f) + R_Z(f)$, and the coherence spectrum,

$$\kappa^2_{X,Y}(f) = \frac{|H(f)|^2 R_X(f)}{|H(f)|^2 R_X(f) + R_Z(f)}.$$

Coherence for filtering and disturbance

Illustration for $R_X(f) = 1$, $R_Z(f) = \sigma^2$ and a lowpass-filter $H(f)$.
Note that the coherence spectrum always is one for $\sigma = 0$.



Estimation of coherence spectrum

If we use the periodogram for estimation of the coherence spectrum

$$\widehat{\kappa^2_{x,y}} = \frac{|\widehat{R}_{x,y}(f)|^2}{\widehat{R}_x(f)\widehat{R}_y(f)} = \frac{\frac{1}{n}\mathcal{X}(f)\mathcal{Y}(f)^* \cdot \frac{1}{n}\mathcal{X}(f)^*\mathcal{Y}(f)}{\frac{1}{n}\mathcal{X}(f)\mathcal{X}(f)^* \cdot \frac{1}{n}\mathcal{Y}(f)\mathcal{Y}(f)^*},$$

the resulting estimate is certainly unreliable although it looks nice!
Use the Welch method or some multitaper approach.