Stationary stochastic processes

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Lecture 5

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Last week

Continuous time: The spectral density function $R(f)$, defined for $-\infty < f < \infty$ is positive, symmetric and integrable such that

$$r(\tau) = \int_{-\infty}^{\infty} R(f)e^{i2\pi f \tau} \, df,$$

and

$$R(f) = \int_{-\infty}^{\infty} r(\tau)e^{-i2\pi f \tau} \, d\tau.$$

The variance is expressed as

$$r_X(0) = \int_{-\infty}^{\infty} R(f) \, df.$$
Last week

Discrete time: The spectral density function $R(f)$, defined for $-1/2 < f \leq 1/2$ is positive, symmetric and integrable such that

$$r(\tau) = \int_{-1/2}^{1/2} R(f)e^{i2\pi f \tau} \, df,$$

and

$$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau)e^{-i2\pi f \tau}.$$

The variance is expressed as

$$r_X(0) = \int_{-1/2}^{1/2} R(f) \, df.$$
Schedule for today

- Sampling in time and frequency
- Aliasing (Vikning)
Example

The random harmonic function \( Y(t) = \cos(2\pi 20t + \phi) \), \( t \in \mathbb{R} \), is sampled with \( t = 0, \pm d, \pm 2d, \ldots \) where \( d = 1/100 \). What is the resulting discrete time sequence, \( Z_n, n = 0, \pm 1, \pm 2, \ldots \)?

Answer: With \( t = nd = n/100 \),

\[
Z_n = Y(nd) = \cos(2\pi \frac{20}{100} n + \phi) = \cos(2\pi \frac{1}{5} n + \phi), \quad n = 0, \pm 1, \pm 2, \ldots
\]
The process $Y(t)$, $t \in \mathbb{R}$, is stationary with covariance function $r_Y(\tau)$, $\tau \in \mathbb{R}$.

Then the sampled sequence $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$, has the same covariance function as $Y(t)$, i.e.,

$$r_Z(\tau) = C[Z_t, Z_{t+\tau}] = C[Y(t), Y(t+\tau)] = r_Y(\tau), \quad \tau = 0, \pm d, \pm 2d, \ldots$$
Reconstruction

Theorem 4.6: If $r_Z(n d)$, $n = 0, \pm 1, \pm 2, \ldots$, is a covariance function in discrete time, then the continuous time covariance function $r_Y(\tau)$, $\tau \in \mathbb{R}$ is equal to the interpolation

$$r_Y(\tau) = \sum_{n=-\infty}^{\infty} r_Z(n d) \frac{\sin \frac{\pi}{d}(\tau - nd)}{\pi \frac{1}{d}(\tau - nd)}.$$

‘Perfect reconstruction’ is not possible in practice, instead often linear or spline interpolation is used.
Sampling-connection in frequency

The sampled covariance function, \( r_Z(\tau) \), \( \tau = 0, \pm d, \pm 2d, \ldots \) is

\[
r_Z(\tau) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f \tau} \, df,
\]

with the sampling frequency \( f_s = 1/d \) and the spectral density is given by

\[
R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) - f_s/2 < f \leq f_s/2.
\]

(Theorem 4.5 with proof p. 97-98.)
Example

The random harmonic function $Y(t) = \cos(2\pi 20t + \phi)$, $t \in \mathbb{R}$, sampled with $t = 0, \pm d, \pm 2d, \ldots$ where $d = 1/100$ is visualized below.
Solution

The random harmonic function \( Y(t) = \cos(2\pi 20t + \phi), \ t \in \mathbb{R}, \) sampled with \( t = 0, \pm d, \pm 2d, \ldots \) where \( d = 1/100 \) is visualized below.

\[
R_Y(f) \\
\sqrt{2} \\
-200 \quad -100 \quad 20 \\
\sqrt{2} \\
-200 \quad -100 \\
\sqrt{2} \\
50 \\
\sqrt{2} \\
100 \quad 200 \\
\sqrt{2} \\
100 \quad 200 \quad 80 \quad 120 \\
\sqrt{2} \\
80 \quad 120 \\
\sqrt{2} \\
20 \quad -120 \quad 80 \quad -200 \\
\sqrt{2} \\
-120 \quad 80 \quad -200 \\
\sqrt{2} \\
-20 \quad 20 \quad -200 \\
\sqrt{2} \\
20 \quad -200 \\
\sqrt{2} \\
-20 \quad 20 \quad -50 \\
\sqrt{2} \\
-50 \quad 20 \\
\sqrt{2} \\
50 \\
\sqrt{2} \\
R_Z(f) \\
\sqrt{2} \\
50 \\
\sqrt{2} \\
20 \quad -50 \\
\sqrt{2} \\
-50 \quad 20 \\
\sqrt{2} \\
-20 \quad 20 \quad -200 \\
\sqrt{2} \\
20 \quad -200 \\
\sqrt{2} \\
-20 \quad 20 \quad -50 \\
\sqrt{2} \\
-50 \quad 20 \\
\sqrt{2} \\
50 \\
\sqrt{2} \\
R_Z(v) \\
\sqrt{2} \\
v = fd \\
-50 \quad 50 \\
-0.5 \quad 0.5 \\
-0.5 \quad 0.5 \\
-0.5 \quad 0.5
Normalized frequency

The sampled covariance function, \( r_Z(\tau) \), \( \tau = 0, \pm d, \pm 2d, \ldots \) where

\[
r_Z(\tau) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f \tau} df,
\]

is converted to the covariance function of a discrete time sequence \( r_X(n) \), \( n = 0, \pm 1, \pm 2, \ldots \), with \( \tau = nd \) and where

\[
r_X(n) = \int_{-1/2}^{1/2} R_X(\nu) e^{i2\pi \nu n} d\nu,
\]

where \( \nu = f \cdot d = f / f_s \) and \( R_X(\nu) = f_s \cdot R_Z(\nu f_s) \).
Example: Normalized frequency

The harmonic function process $Y(t) = \cos(2\pi 20t + \phi)$, $t \in \mathbb{R}$, is sampled with $d = 1/f_s = 1/100$. 

![Graph showing normalized frequency spectrum with points at f/Hz and v=fd axes]
Sampling and aliasing

Example: The realizations $x(t) = \cos(2\pi 10t)$ and $w(t) = \cos(2\pi 50t)$, $t \in \mathbb{R}$, is sampled at $t = nd$, $n = 0, \pm 1, \pm 2, \ldots$ with $d = 1/f_s = 1/40$, 

$$x_n = x(nd) = \cos(2\pi \frac{10}{40} n) = \cos(2\pi \frac{1}{4} n).$$

and

$$w_n = \cos(2\pi \frac{50}{40} n) = \cos(2\pi \frac{5}{4} n) = \cos(2\pi (\frac{1}{4} + 1)n) = \cos(2\pi \frac{1}{4} n) = x_n.$$ 

The realization $w_n$ is aliased to become the realization $x_n$. 
Sampling and aliasing

A high frequency signal will be interpreted as a low frequency signal if the sampling interval is too large and the resulting sampling frequency too small.
Sampling and aliasing

Harmonic function realizations of different frequencies $f_0$, are sampled with $f_s = 40$, giving

$$x(t) = \cos(2\pi 10t) \quad \rightarrow \quad x_n = \cos(2\pi \frac{10}{40} n) = \cos(2\pi \frac{1}{4} n),$$

$$y(t) = \cos(2\pi 20t) \quad \rightarrow \quad y_n = \cos(2\pi \frac{20}{40} n) = \cos(2\pi \frac{1}{2} n) = (-1)^n,$$

$$v(t) = \cos(2\pi 40t) \quad \rightarrow \quad v_n = \cos(2\pi \frac{40}{40} n) = \cos(2\pi n) = 1,$$

$$w(t) = \cos(2\pi 50t) \quad \rightarrow \quad w_n = \cos(2\pi \frac{50}{40} n) = \cos(2\pi \frac{1}{4} n).$$
Example: Aliasing

The harmonic function process \( Y(t) = \cos(2\pi 60t + \phi) \), \( t \in \mathbb{R} \), is sampled with \( d = 1/f_s = 1/100 \).
The sampling theorem

The sampling frequency should be

\[ f_s \geq 2f_{\text{max}}, \]

to avoid aliasing.

The Nyquist frequency

\[ f_n = \frac{f_s}{2}. \]

is the maximum possible frequency after sampling.

The sampling theorem is often referred to as the Nyquist-Shannon sampling theorem.
Solution: Aliasing

The harmonic function process \( Y(t) = \cos(2\pi 60t + \phi), \ t \in \mathbb{R} \), is sampled with \( d = 1/f_s = 1/100 \).

The resulting spectral density corresponds to a harmonic function process \( X(t) = \cos(2\pi 40t + \phi), \ t \in \mathbb{R} \).
Exam exercise

A continuous time stationary stochastic process $X(t)$, $t \in \mathbb{R}$, is described by the spectral density according to the figure below. The process is sampled with the sampling distance $d = 1/40$. Determine the spectral density for the sampled process. The answer should be motivated by calculations or figures.
Solution: Exam exercise

The solution can be found according to the following figures, where different colors show the different frequency-shifted continuous time spectral densities.
Another exam exercise

We now define the stationary process $Y(t) = m + X(t)$, $t \in \mathbb{R}$, where $m$ is unknown. The process $X(t)$ is assumed to have $E[X(t)] = 0$ and spectral density as defined in the previous example. An estimate of $m$ should be found as

$$
\hat{m}_n = \frac{Y(d) + Y(2d) + \ldots + Y(nd)}{n},
$$

by sampling the process $Y(t)$ with sample distance $d = 1/40$.

a) Determine the spectral density of the sampled process $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$ for the sample distance $d = 1/40$.

b) Determine the variance, $V[\hat{m}_n]$, when $d = 1/40$. 
Solution: Another exam exercise

a) The sampling frequency $f_s = 40$ will cause aliasing and the resulting spectral density will be constant with $R_Z(f) = 3, -20 < f \leq 20$.

b) We find

$$r_Z(\tau) = \begin{cases} 
120 & \tau = 0 \\
0 & \tau = \pm 1, \pm 2, \ldots
\end{cases}$$

The variance will be

$$V[\hat{m}_n] = \frac{nV[Z_t]}{n^2} = \frac{120}{n}.$$