Stationary stochastic processes

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Lecture 5

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Last week

Continuous time: The spectral density function $R(f)$, defined for $-\infty < f < \infty$ is **positive**, **symmetric** and **integrable** such that

$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f \tau} df,$$

and

$$R(f) = \int_{-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau} d\tau.$$

The variance is expressed as

$$r_X(0) = \int_{-\infty}^{\infty} R(f) df.$$

Last week

Discrete time: The spectral density function $R(f)$, defined for $-1/2 < f \leq 1/2$ is **positive**, **symmetric** and **integrable** such that

$$r(\tau) = \int_{-1/2}^{1/2} R(f) e^{i2\pi f \tau} df,$$

and

$$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau}.$$

The variance is expressed as

$$r_X(0) = \int_{-1/2}^{1/2} R(f) df.$$

Schedule for today

- Sampling in time and frequency
- Aliasing (Vikning)

Example

The random harmonic function $Y(t) = \cos(2\pi 20t + \phi)$, $t \in \mathbb{R}$, is sampled with $t = 0, \pm d, \pm 2d, \dots$ where $d = 1/100$. What is the resulting discrete time sequence, Z_n, $n = 0, \pm 1, \pm 2, \dots$?

Answer: With $t = nd = n/100$,

$$Z_n = Y(nd) = \cos(2\pi \frac{20}{100} n + \phi) = \cos(2\pi \frac{1}{5} n + \phi), \quad n = 0, \pm 1, \pm 2, \dots.$$

Sampling-connection in time

The process $Y(t)$, $t \in \mathbb{R}$, is stationary with covariance function $r_Y(\tau)$, $\tau \in \mathbb{R}$.

Then the sampled sequence $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$, has the same covariance function as $Y(t)$, i.e.,

$$r_Z(\tau) = C[Z_t, Z_{t+\tau}] = C[Y(t), Y(t+\tau)] = r_Y(\tau), \quad \tau = 0, \pm d, \pm 2d, \ldots$$

Reconstruction

Theorem 4.6: If $r_Z(nd)$, $n = 0, \pm 1, \pm 2, \ldots$, is a covariance function in discrete time, then the continuous time covariance function $r_Y(\tau)$, $\tau \in \mathbb{R}$ is equal to the interpolation

$$r_Y(\tau) = \sum_{n=-\infty}^{\infty} r_Z(nd) \frac{\sin \pi \frac{1}{d}(\tau - nd)}{\pi \frac{1}{d}(\tau - nd)}.$$

'Perfect reconstruction' is not possible in practice, instead often linear or spline interpolation is used.

Sampling-connection in frequency

The sampled covariance function, $r_Z(\tau)$, $\tau = 0, \pm d, \pm 2d, \ldots$ is

$$r_Z(\tau) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f \tau} df,$$

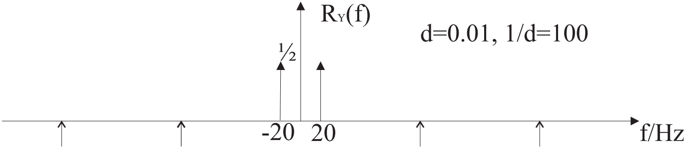
with the sampling frequency $f_s = 1/d$ and the spectral density is given by

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) \quad -f_s/2 < f \leq f_s/2.$$

(Theorem 4.5 with proof p. 97-98.)

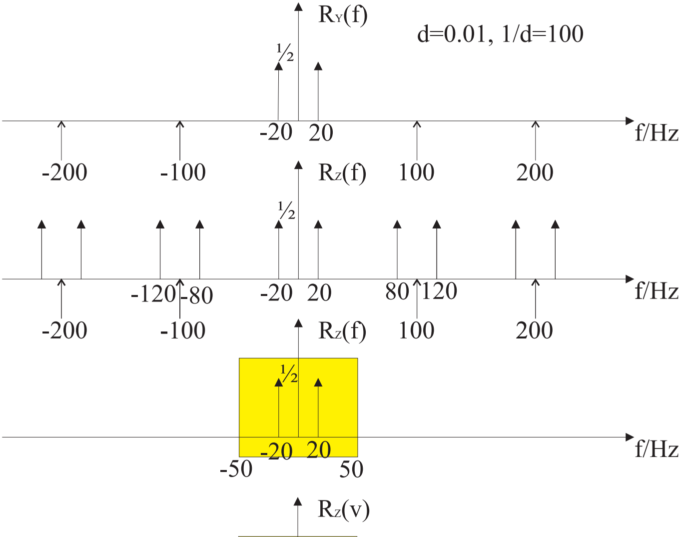
Example

The random harmonic function $Y(t) = \cos(2\pi 20t + \phi)$, $t \in \mathbb{R}$, sampled with $t = 0, \pm d, \pm 2d, \ldots$ where $d = 1/100$ is visualized below.



Solution

The random harmonic function $Y(t) = \cos(2\pi 20t + \phi)$, $t \in \mathbb{R}$, sampled with $t = 0, \pm d, \pm 2d, \ldots$ where $d = 1/100$ is visualized below.



Normalized frequency

The sampled covariance function, $r_Z(\tau)$, $\tau = 0, \pm d, \pm 2d, \ldots$ where

$$r_Z(\tau) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f \tau} df,$$

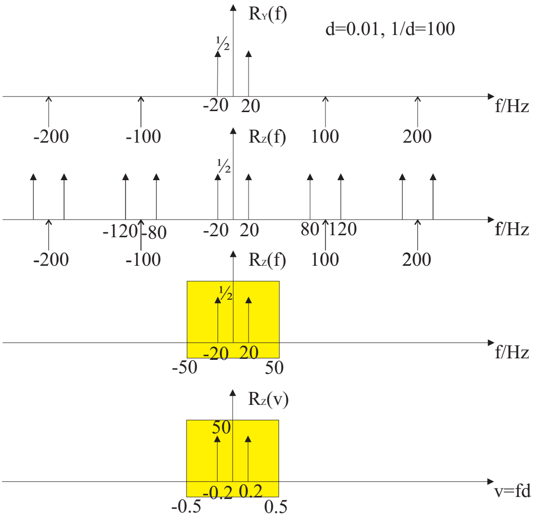
is converted to the covariance function of a discrete time sequence $r_X(n)$, $n = 0, \pm 1, \pm 2, \ldots$, with $\tau = nd$ and where

$$r_X(n) = \int_{-1/2}^{1/2} R_X(\nu) e^{i2\pi \nu n} d\nu,$$

where $\nu = f \cdot d = f/f_s$ and $R_X(\nu) = f_s \cdot R_Z(\nu f_s)$.

Example: Normalized frequency

The harmonic function process $Y(t) = \cos(2\pi 20t + \phi)$, $t \in \mathbb{R}$, is sampled with $d = 1/f_s = 1/100$.



Sampling and aliasing

Example: The realizations $x(t) = \cos(2\pi 10t)$ and $w(t) = \cos(2\pi 50t)$, $t \in \mathbb{R}$, is sampled at $t = nd$, $n = 0, \pm1, \pm2, \ldots$ with $d = 1/f_s = 1/40$,

$$x_n = x(nd) = \cos(2\pi \frac{10}{40} n) = \cos(2\pi \frac{1}{4} n).$$

and

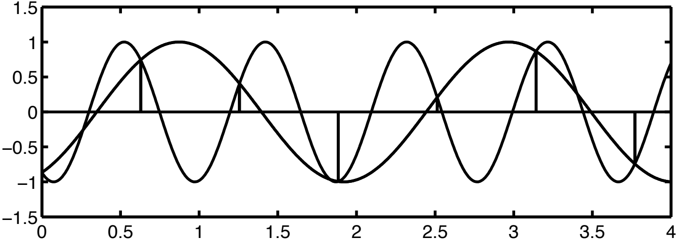
$$w_n = \cos(2\pi \frac{50}{40} n) = \cos(2\pi \frac{5}{4} n) =$$

$$\cos(2\pi(\frac{1}{4} + 1)n) = \cos(2\pi \frac{1}{4} n) = x_n.$$

The realization w_n is aliased to become the realization x_n.

Sampling and aliasing

A high frequency signal will be interpreted as a low frequency signal if the sampling interval is too large and the resulting sampling frequency too small.



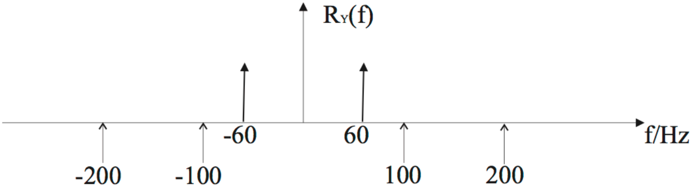
Sampling and aliasing

Harmonic function realizations of different frequencies f_0, are sampled with $f_s = 40$, giving

$$
\begin{aligned}
x(t) = \cos(2\pi 10t) \quad &\rightarrow \quad x_n = \cos(2\pi \frac{10}{40} n) = \cos(2\pi \frac{1}{4} n), \\
y(t) = \cos(2\pi 20t) \quad &\rightarrow \quad y_n = \cos(2\pi \frac{20}{40} n) = \cos(2\pi \frac{1}{2} n) = (-1)^n, \\
v(t) = \cos(2\pi 40t) \quad &\rightarrow \quad v_n = \cos(2\pi \frac{40}{40} n) = \cos(2\pi n) = 1, \\
w(t) = \cos(2\pi 50t) \quad &\rightarrow \quad w_n = \cos(2\pi \frac{50}{40} n) = \cos(2\pi \frac{1}{4} n).
\end{aligned}
$$

Example: Aliasing

The harmonic function process $Y(t) = \cos(2\pi 60t + \phi)$, $t \in \mathbb{R}$, is sampled with $d = 1/f_s = 1/100$.



The sampling theorem

The sampling frequency should be

$$f_s \geq 2f_{max},$$

to avoid aliasing.

The Nyquist frequency

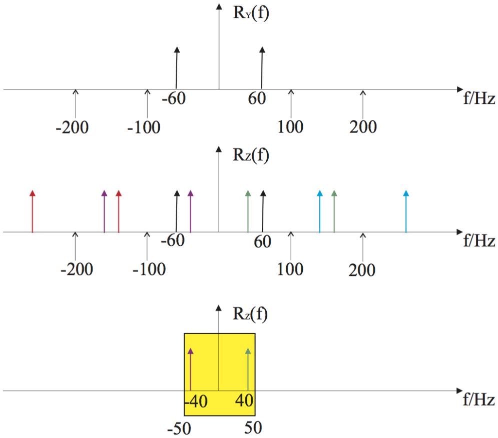
$$f_n = \frac{f_s}{2}.$$

is the maximum possible frequency after sampling.

The sampling theorem is often referred to as the Nyquist-Shannon sampling theorem.

Solution: Aliasing

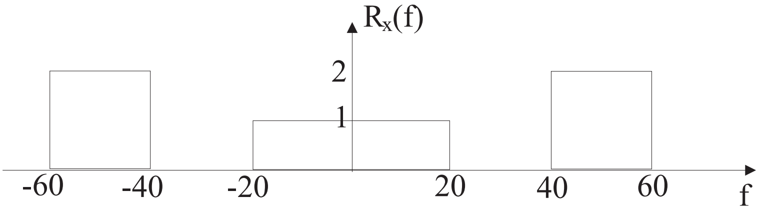
The harmonic function process $Y(t) = \cos(2\pi 60t + \phi)$, $t \in \mathbb{R}$, is sampled with $d = 1/f_s = 1/100$.



The resulting spectral density corresponds to a harmonic function process $X(t) = \cos(2\pi 40t + \phi)$, $t \in \mathbb{R}$.

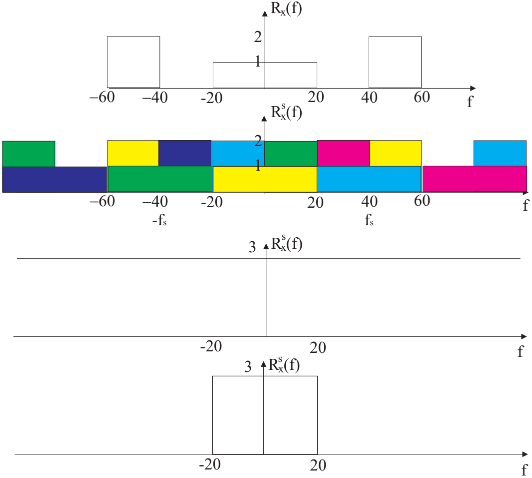
Exam exercise

A continuous time stationary stochastic process $X(t)$, $t \in \mathbb{R}$, is described by the spectral density according to the figure below. The process is sampled with the sampling distance $d = 1/40$. Determine the spectral density for the sampled process. The answer should be motivated by calculations or figures.



Solution: Exam exercise

The solution can be found according to the following figures, where different colors shows the different frequency-shifted continuous time spectral densities.



Another exam exercise

We now define the stationary process $Y(t) = m + X(t)$, $t \in \mathbb{R}$, where m is unknown. The process $X(t)$ is assumed to have $E[X(t)] = 0$ and spectral density as defined in the previous example. An estimate of m should be found as

$$\widehat{m}_n = \frac{Y(d) + Y(2d) + \ldots Y(nd)}{n},$$

by sampling the process $Y(t)$ with sample distance $d = 1/40$.

a) Determine the spectral density of the sampled process $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$ for the sample distance $d = 1/40$.

b) Determine the variance, $V[\widehat{m}_n]$, when $d = 1/40$.

Solution: Another exam exercise

a) The sampling frequency $f_s = 40$ will cause aliasing and the resulting spectral density will be constant with $R_Z(f) = 3$, $-20 < f \leq 20$.

b) We find

$$r_Z(\tau) = \begin{cases} 120 & \tau = 0 \\ 0 & \tau = \pm 1, \pm 2, \ldots. \end{cases}$$

The variance will be

$$V[\widehat{m}_n] = \frac{nV[Z_t]}{n^2} = \frac{120}{n}.$$