Stationary stochastic processes

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Lecture 5

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Cont. time and spectral density

Continuous time: The spectral density function $R(f)$, defined for $-\infty < f < \infty$ is **positive**, **symmetric** and **integrable** such that

$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f \tau} df,$$

and

$$R(f) = \int_{-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau} d\tau.$$

The variance is expressed as

$$r_X(0) = \int_{-\infty}^{\infty} R(f) df.$$

Discrete time and spectral density

Discrete time: The spectral density function $R(f)$, defined for $-1/2 < f \leq 1/2$ is **positive**, **symmetric** and **integrable** such that

$$r(\tau) = \int_{-1/2}^{1/2} R(f) e^{i2\pi f \tau} df,$$

and

$$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau}.$$

The variance is expressed as

$$r_X(0) = \int_{-1/2}^{1/2} R(f) df.$$

Schedule for today

- Sampling in time and frequency
- Aliasing

Sampling-connection in time

The process $Y(t)$, $t \in \mathbb{R}$, is stationary with covariance function $r_Y(\tau)$, $\tau \in \mathbb{R}$.

Then the sampled sequence $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$, has the same covariance function as $Y(t)$, i.e.,

$$r_Z(\tau) = C[Z_t, Z_{t+\tau}] = C[Y(t), Y(t+\tau)] = r_Y(\tau), \;\; \tau = 0, \pm d, \pm 2d, \ldots$$

at the same sample distance d related to the sampling frequency as $f_s = 1/d$.

Reconstruction

Theorem 4.6: If $r_Z(nd)$, $n = 0, \pm 1, \pm 2, \ldots$, is a covariance function in discrete-time, then the continuous time covariance function $r_Y(\tau)$, $\tau \in \mathbb{R}$ is equal to the interpolation

$$r_Y(\tau) = \sum_{n=-\infty}^{\infty} r_Z(nd) \frac{\sin \pi f_s(\tau - nd)}{\pi f_s(\tau - nd)}.$$

'Perfect reconstruction' is not possible in practice, instead often linear or spline interpolation is used.

Sampling-connection in frequency

The sampled covariance function, $r_Z(\tau)$, $\tau = 0, \pm d, \pm 2d, \ldots$ is

$$r_Z(\tau) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f \tau} df,$$

with $f_s = 1/d$ and the spectral density is given by

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) \quad -f_s/2 < f \leq f_s/2,$$

where

$$R_Y(f) = \int_{-\infty}^{\infty} r_Y(\tau) e^{-i2\pi f \tau} d\tau.$$

(Theorem 4.5 with proof p. 97-98.)

Normalized frequency

The sampled covariance function, $r_Z(\tau)$, $\tau = 0, \pm d, \pm 2d, \ldots$ where

$$r_Z(\tau) = \int_{-f_s/2}^{f_s/2} R_Z(f) e^{i2\pi f \tau} df,$$

is converted to the covariance function of a discrete-time sequence $r_X(n)$, $n = 0, \pm 1, \pm 2, \ldots$, with $\tau = nd$ and where

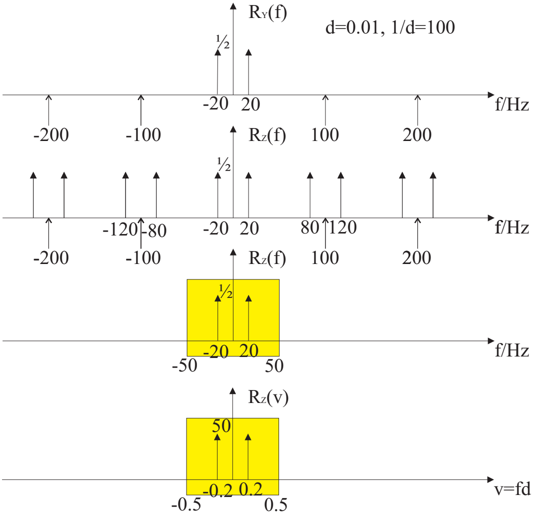
$$r_X(n) = \int_{-1/2}^{1/2} R_X(\nu) e^{i2\pi \nu n} d\nu,$$

where $\nu = f \cdot d = f/f_s$ and $R_X(\nu) = f_s \cdot R_Z(\nu f_s)$.

Example

The random harmonic function $Y(t) = \cos(2\pi 20t + \phi)$, $t \in \mathbb{R}$, is sampled with $f_s = 1/d = 100$. What is the resulting discrete-time sequence, Z_n, $n = 0, \pm 1, \pm 2, \ldots$?

Example



RY(f)
d=0.01, 1/d=100
½
-20
20
f/Hz
-200
-100
100
200
RZ(f)
½
-120
-80
-20
20
80
120
f/Hz
-200
-100
100
200
RZ(f)
½
-20
20
f/Hz
-50
50
RZ(v)
50
-0.2
0.2
v=fd
-0.5
0.5

Sampling and aliasing

Example: The realizations $x(t) = \cos(2\pi 10t)$ and $w(t) = \cos(2\pi 50t)$, $t \in \mathbb{R}$, is sampled at $t = nd$, $n = 0, \pm 1, \pm 2, \ldots$ with $d = 1/f_s = 1/40$,

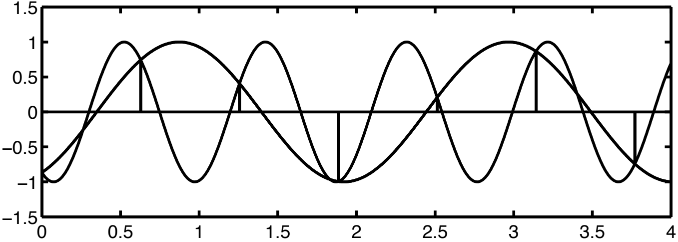
$$x_n = x(nd) = \cos(2\pi \frac{10}{40} n) = \cos(2\pi \frac{1}{4} n).$$

and

$$w_n = \cos(2\pi \frac{50}{40} n) = \cos(2\pi \frac{5}{4} n) = \cos(2\pi (\frac{1}{4} + 1) n) = \cos(2\pi \frac{1}{4} n) = x_n.$$

Sampling and aliasing

A high frequency signal will be interpreted as a low frequency signal if the sampling interval is too large and the resulting sampling frequency too small.



Sampling and aliasing

Realizations $\cos(2\pi f_0 t)$, $t \in \mathbb{R}$, of different frequencies f_0, are sampled with $f_s = 40$, giving

$$
\begin{aligned}
x(t) = \cos(2\pi 10 t) \quad &\rightarrow \quad x_n = \cos(2\pi \frac{10}{40} n) = \cos(2\pi \frac{1}{4} n), \\
y(t) = \cos(2\pi 20 t) \quad &\rightarrow \quad y_n = \cos(2\pi \frac{20}{40} n) = \cos(2\pi \frac{1}{2} n) = (-1)^n, \\
v(t) = \cos(2\pi 40 t) \quad &\rightarrow \quad v_n = \cos(2\pi \frac{40}{40} n) = \cos(2\pi n) = 1, \\
w(t) = \cos(2\pi 50 t) \quad &\rightarrow \quad w_n = \cos(2\pi \frac{50}{40} n) = \cos(2\pi \frac{1}{4} n).
\end{aligned}
$$

The sampling theorem

The sampling frequency should be

$$f_s \geq 2f_{max},$$

to avoid aliasing. The Nyquist frequency is the maximum possible frequency after sampling,

$$f_n = \frac{f_s}{2}.$$

The sampling theorem is often referred to as the Nyquist-Shannon sampling theorem but many other researchers had similar ideas.

Old exam (modified)

A stationary continuous-time process $X(t),\ t \in \mathbb{R}$, has the spectral density

$$R_X(f) = \begin{cases} 1 - |f| & |f| \leq 1 \\ 0 & |f| > 1. \end{cases}$$

a) Determine the covariance function of the process.

b) The process $X(t)$ is sampled with $f_s = 3$, giving a the discrete-time process $Y_n,\ n = 0 \pm 1, \pm 2, \ldots$, where $n = t/d$. Determine the spectral density for Y_n.

c) The process $X(t)$ is now sampled with $f_s = 3/2$ and a new process $Z_t,\ t = 0 \pm d, \pm 2d, \ldots$, is computed. Determine the spectral density for the sampled process Z_t.

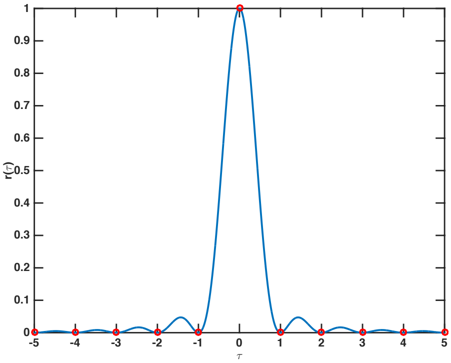
d) Determine the covariance function of the sampled process W_n, $n = 0 \pm 1, \pm 2, \ldots$, where $n = t/d$, if instead $X(t)$ is sampled with $f_s = 1$.

a) Table of formulas

$\begin{cases} \alpha & \text{if} \quad \tau = 0 \\ \frac{\sin(2\pi\alpha\tau)}{2\pi\tau} & \text{if} \quad \tau \neq 0 \end{cases}$	$\begin{cases} 1/2 & \text{if} \quad \lvert f \rvert \leq \alpha \\ 0 & \text{if} \quad \lvert f \rvert > \alpha \end{cases}$
$\begin{cases} 1 - \alpha\lvert\tau\rvert & \text{if} \quad \lvert\tau\rvert \leq \frac{1}{\alpha} \\ 0 & \text{if} \quad \lvert\tau\rvert > \frac{1}{\alpha} \end{cases}$	$\begin{cases} \frac{1}{\alpha} & \text{if} \quad f = 0 \\ \frac{2\alpha}{(2\pi f)^2}\left(1 - \cos\left(\frac{2\pi f}{\alpha}\right)\right) & \text{if} \quad f \neq 0 \end{cases}$
$g(\tau)h(\tau)$	$G(f) * H(f) = \int G(\nu)H(f-\nu)d\nu$
$g(\tau) * h(\tau) = \int g(t)h(\tau - t)dt$	$G(f)H(f)$

d) Covariance function

Blue line shows $r_X(\tau)$, $\tau \in \mathbb{R}$ and red circles represent $r_W(\tau)$, $\tau = 0, \pm 1, \pm 2, \ldots$ of the sampled sequence.



d) Covariance function

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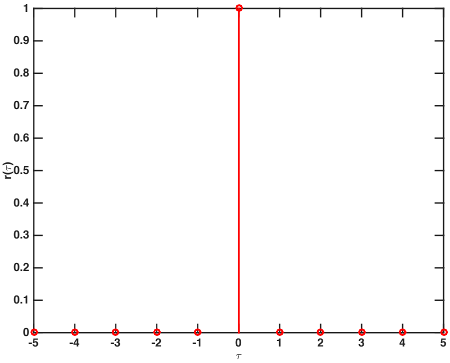
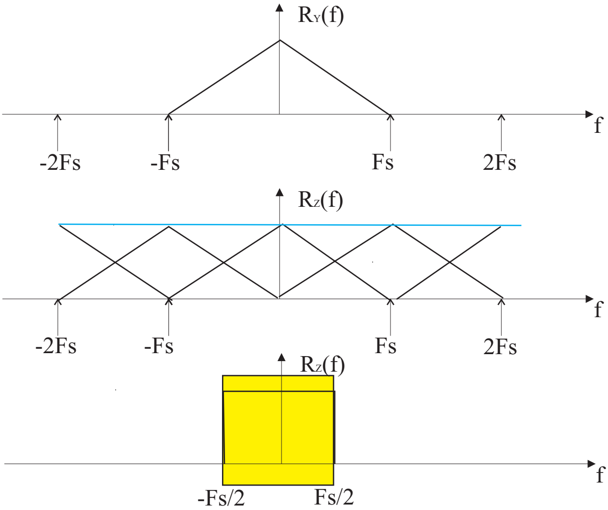


d) Spectral density

To low value of f_s causes aliasing to a discrete-time white noise sequence.



Solution

a) The covariance function is

$$r_X(\tau) = \begin{cases} 1 & \tau = 0 \\ \frac{2}{(2\pi\tau)^2}(1 - \cos(2\pi\tau)) = \frac{1}{(\pi\tau)^2}\sin^2(\pi\tau) & \tau \neq 0. \end{cases}$$

b) The spectral density is

$$R_Y(\nu) = \begin{cases} 3(1 - 3|\nu|) & |\nu| \leq \frac{1}{3} \\ 0 & \frac{1}{3} < |\nu| \leq \frac{1}{2}. \end{cases}$$

c) The spectral density is

$$R_Z(f) = \begin{cases} 1 - |f| & |f| \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < |f| \leq \frac{3}{4}. \end{cases}$$

d) The covariance function is

$$r_W(\tau) = \begin{cases} 1 & \tau = 0 \\ 0 & \tau \neq 0. \end{cases}$$