Stationary stochastic processes

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Lecture 4

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Yesterday news

A stationary Gaussian process is strictly (and weakly) stationary as the specification of m_x and $r_X(0)$ also defines the density function,

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi r_X(0)}} e^{-(x-m_X)^2/2r_X(0)}.$$

Linear operations applied to Gaussian processes such as e.g., addition, subtraction, differentiation and integration, result in Gaussian processes.

Covariance function

The random harmonic function,

$$X(t) = A\cos(2\pi f_0 t + \phi), \quad t \in \mathbb{R},$$

with $\phi \in U(0, 2\pi)$ and A as independent stochastic variables, is a strictly stationary process. The mean value is $m_X = 0$, the covariance function is found as

$$r_X(\tau) = \frac{E[A^2]}{2}\cos(2\pi f_0 \tau), \quad \tau \in \mathbb{R},$$

and the variance is $r_X(0) = \frac{E[A^2]}{2}$. The spectral density is

$$R(f) = \frac{E[A^2]}{4}(\delta(f - f_0) + \delta(f + f_0)),$$

where $\delta(f)$ is the Dirac delta function, i.e. the variance is equally spread on $f = f_0$ and $-f_0$.

Schedule for today

- Spectral density in continuous time
- Calculation and interpretation of spectral densities
- Spectral density in discrete time
- The periodogram

Spectral density

Theorem 4.1: If the covariance function $r(\tau)$ of a stationary process $X(t), t \in \mathbb{R}$ is continuous, there exists a **positive**, **symmetric** and **integrable** spectral density function (spektraltäthet) $R(f)$ such that

$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f \tau} df,$$

and

$$R(f) = \int_{-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau} d\tau.$$

The covariance function and the spectral density are real-valued for real-valued processes.

The variance is given as

$$r(0) = \int_{-\infty}^{\infty} R(f) df.$$

Example: Low-frequency noise

For a low-frequency noise spectral density, $R(f) = 1/2$ for $-1 \leq f \leq 1$ and zero for all other values.

The covariance function is given by

$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f \tau} df = \frac{1}{2} \int_{-1}^{1} e^{i2\pi f \tau} df = \frac{\sin(2\pi\tau)}{2\pi\tau}.$$

The table of formulas

Someone else has computed what we need...

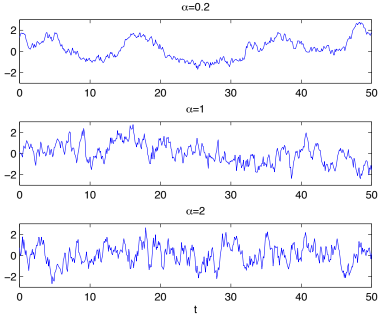
Fourier transforms

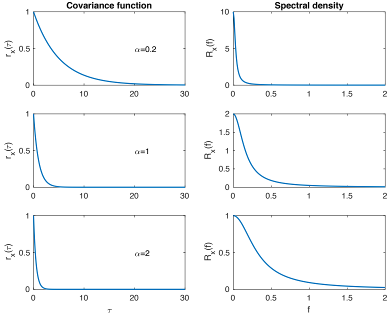
$g(\tau) \qquad (\alpha > 0)$	$G(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} g(\tau)\, d\tau$
$e^{-\alpha\lvert\tau\rvert}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$\frac{1}{\alpha^2+\tau^2}$	$\frac{\pi}{\alpha}\, e^{-2\pi\alpha\lvert f\rvert}$
$\lvert\tau\rvert e^{-\alpha\lvert\tau\rvert}$	$2\,\frac{(\alpha^2-(2\pi f)^2)}{(\alpha^2+(2\pi f)^2)^2}$
$\lvert\tau\rvert^k e^{-\alpha\lvert\tau\rvert}$	$\frac{k!}{(\alpha^2+(2\pi f)^2)^{k+1}}\{(\alpha+i2\pi f)^{k+1}+(\alpha-i2\pi f)^{k+1}\}$
$e^{-\alpha\tau^2}$	$\sqrt{\pi/\alpha}\exp(-\frac{(2\pi f)^2}{4\alpha})$
$e^{-\alpha\lvert\tau\rvert}\cos(2\pi f_0\tau)$	$\frac{\alpha}{\alpha^2+(2\pi f_0-2\pi f)^2}+\frac{\alpha}{\alpha^2+(2\pi f_0+2\pi f)^2}$

Example: The Ornstein-Uhlenbeck process

For the Ornstein-Uhlenbeck process the covariance function is $r(\tau) = e^{-\alpha|\tau|}$ and the spectral density $R(f) = \frac{2\alpha}{\alpha^2+(2\pi f)^2}$.

Examples:

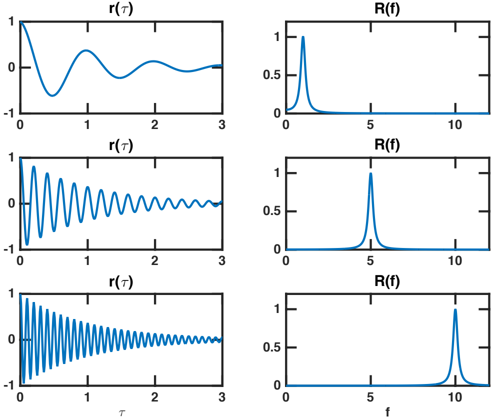




Frequency and period time

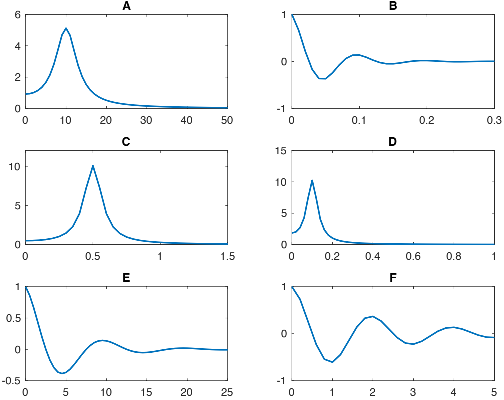
The period time is related to the frequency as $T = 1/f_0$.

	$e^{-\alpha|\tau|}\cos(2\pi f_0 \tau)$	$\frac{\alpha}{\alpha^2+(2\pi f_0-2\pi f)^2} + \frac{\alpha}{\alpha^2+(2\pi f_0+2\pi f)^2}$



Example

Determine the three figures that are spectral densities and the three that are covariance functions. Combine the corresponding covariance function and spectral density.



A computational trick

Fourier transforms

$g(\tau) \qquad (\alpha > 0)$	$G(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} g(\tau)\, d\tau$
$e^{-\alpha\lvert\tau\rvert}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$\frac{1}{\alpha^2+\tau^2}$	$\frac{\pi}{\alpha} e^{-2\pi\alpha\lvert f\rvert}$

Example: Calculate the corresponding covariance function of the spectral density

$$R(f) = \frac{1}{\beta^2 + f^2}.$$

A computational trick

Solution 2: Using the positive symmetry properties of the covariance function and spectral density,

$$R(f) = \int_{-\infty}^{\infty} r(\tau)e^{-i2\pi f\tau}d\tau = 2\int_0^{\infty} r(\tau)\cos(2\pi f\tau)d\tau, \quad r(\tau) = \int_{-\infty}^{\infty} R(f)e^{i2\pi f\tau}df = 2\int_0^{\infty} R(f)\cos(2\pi f\tau)df,$$

allow us to use the table of Fourier transforms more extensively. From the table of formulas we find

$\frac{1}{\alpha^2+\tau^2}$	$\frac{\pi}{\alpha}e^{-2\pi\alpha\|f\|}$

where we switch τ for f, resulting in

$\frac{\pi}{\alpha}e^{-2\pi\alpha\|\tau\|}$	$\frac{1}{\alpha^2+f^2}$.

Replacing $\alpha = \beta$, the solution is $r(\tau) = \frac{\pi}{\beta}e^{-2\pi\beta|\tau|}$ without any calculations at all.

Convince yourself by studying Example 3.pdf at the webpage.

Discrete time and spectral density

Theorem 4.4: For every covariance function $r(\tau)$ of a stationary sequence, $X_t, t = 0, \pm 1, \pm 2, \ldots$, there exists a positive, symmetric and integrable density function $R(f)$ defined for $-1/2 < f \leq 1/2$ such as

$$r(\tau) = \int_{-1/2}^{1/2} R(f) e^{i2\pi f \tau} df.$$

The spectral density is found as

$$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau}.$$

Example: White noise in discrete time

For an uncorrelated sequence (white noise) in discrete time the covariance function is

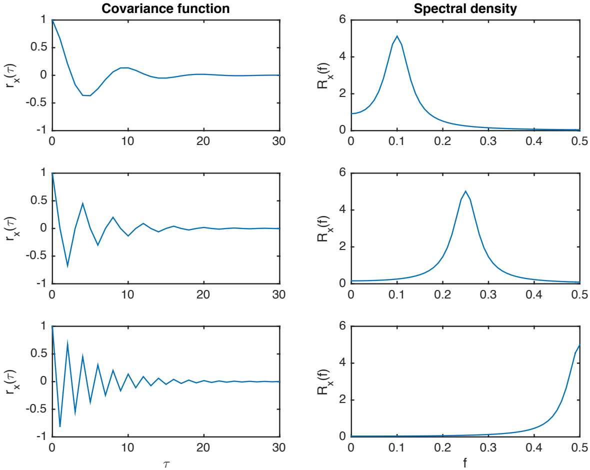
$$r(\tau) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = \pm 1, \pm 2, \ldots \end{cases}$$

where we can also write $r(\tau) = \sigma^2 \delta(\tau)$ with the Kronecker delta function defined as $\delta(\tau) = 1$ for $\tau = 0$ and zero for $\tau \neq 0$.
We find the spectral density as

$$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau} = r(0) = \sigma^2, \;\; -1/2 < f \leq 1/2.$$

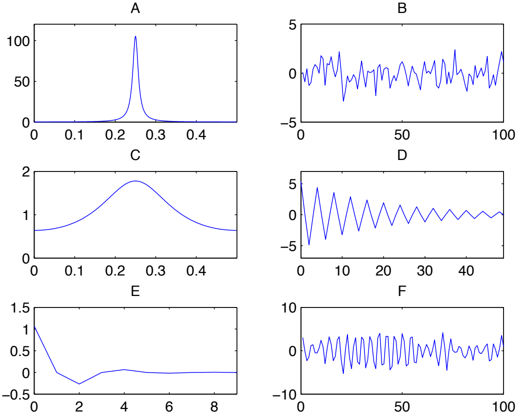
Example

Increasing frequency to highest possible $f_0 = 0.5$.



Exam exercise

The following figures show realizations, covariance functions and spectral densities of two discrete-time processes. Determine and motivate, what is shown in each figure. Also state which realization, covariance function and spectral density that belong to each of the two processes.



Old exam exercise

Let e_t, $t = 0, \pm 1, \pm 2, \ldots$, be a discrete time uncorrelated sequence with $e_t \in N(0, 1)$. Define a new stochastic process X_t by the moving average

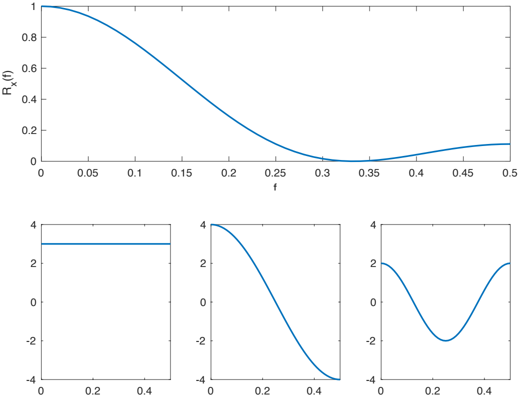
$$X_t = \frac{e_t + e_{t-1} + e_{t-2}}{3}.$$

Calculate the spectral density of the process X_t.

Solution: old exam exercise

The covariance is $r_X(0) = \frac{3}{9}$, $r_X(\pm 1) = \frac{2}{9}$, $r_X(\pm 2) = \frac{1}{9}$, and zero for all other values. The spectral density becomes

$$R_X(f) = \frac{3}{9} + \frac{4}{9}\cos(2\pi f) + \frac{2}{9}\cos(4\pi f).$$



Estimation of spectral density

Let $x_t, t = 0, 1, 2, \ldots n - 1$ be a sequence of data. Compute the Fourier transform,

$$\mathcal{X}(f) = \sum_{t=0}^{n-1} x_t e^{-i2\pi ft}.$$

The *periodogram* is defined as

$$\widehat{R}_x(f) = \frac{1}{n} |\mathcal{X}(f)|^2,$$

and is an estimate of the spectral density.

The periodogram was invented by Sir Arthur Schuster already in 1898, by see the famous paper - "On the investigation of hidden periodicities with application to a supposed 26 day period of meteorological phenomena," Terrestrial Magnetism, 3, 13-41, 1898.

The DFT and FFT

With the Discrete Fourier Transform (DFT),

$$\mathcal{X}(\frac{k}{N}) = \sum_{t=0}^{n-1} x_t e^{-i2\pi \frac{k}{N} t},$$

where $k = 0, 1, \ldots, N-1$, the periodogram can be computed as,

$$\widehat{R}_x(\frac{k}{N}) = \frac{1}{n} |\mathcal{X}(\frac{k}{N})|^2,$$

The calculations are often made using the Fast Fourier Transform (FFT) algorithm, with $N = 2^i$ for any integer i where $N \geq n$.

Example

The DFT of a realization of the random harmonic function $X_t = A\cos(2\pi \frac{k_0}{N} t + \phi)$, $t = 0, 1, 2, \ldots N-1$, where $k_0 < N/2$ is

$$\mathcal{X}(\frac{k}{N}) = \frac{A \cdot N}{2} e^{i\phi}, \quad k = k_0,$$

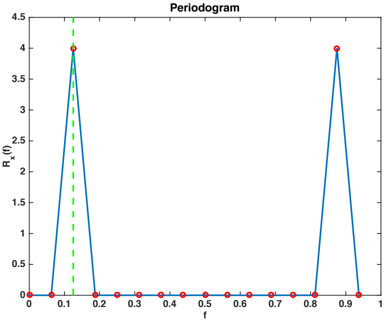
and zero for other values $0 \leq k \leq N/2 - 1$. The estimated spectral density is

$$\widehat{R}_x(\frac{k}{N}) = \frac{1}{N} |\mathcal{X}(\frac{k}{N})|^2 = \frac{A^2 \cdot N}{4}, \quad k = k_0,$$

and zero for other values $0 \leq k \leq N/2 - 1$. For proof see section 4.3.2 p.95.

Example

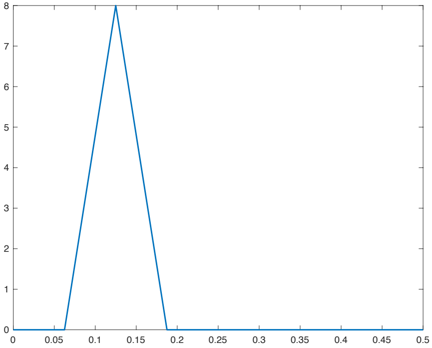
The periodogram of a discrete time sequence $x_t = \cos(2\pi f_0 t)$, $t = 0, 1, \ldots, N-1$, with frequency $f_0 = 0.125$ and $N = 16$ is calculated. The result is shown where the frequencies $-0.5 < f < 0$ are now found $0.5 < f < 1$.




```
x=cos(2*pi*f*[0:N-1]');
X=fft(x);
stem([0:N-1]/N,1/N*abs(X).^ 2);
```

Example

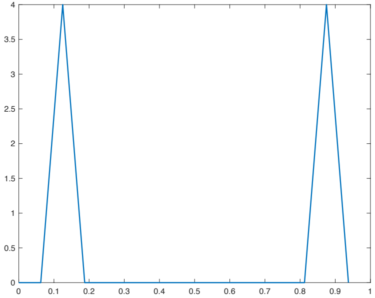
W often use the one-sided periodogram. Note that in Matlab all power is moved to the positive side.




```
[P,f]=periodogram(x,[],16,1);
plot(f,P)
```

Example

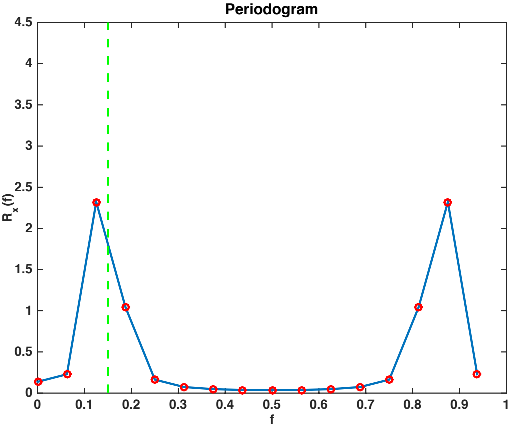
If we instead use the two-sided the power is equally spread at the two frequencies.




```
[P,f]=periodogram(x,[],'twosided',16,1);
plot(f,P)
```

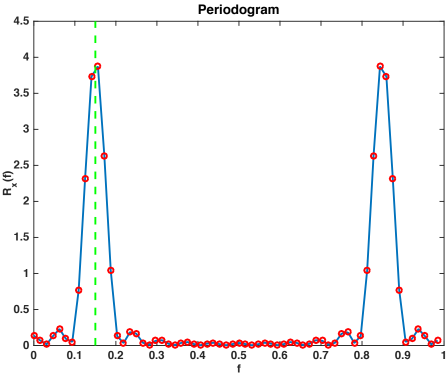
Example: A cosine

If the frequency of the cosine is slightly changed, e.g., to $f_0 = 0.15$, the view changes considerably.



Example: A cosine

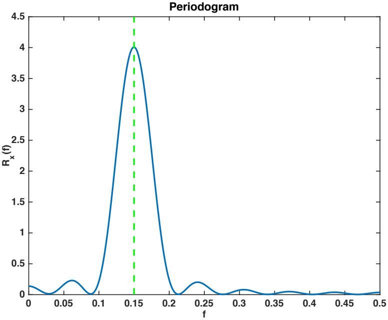
We use zero-padding (noll-utfyllnad), i.e. computing the periodogram at $N = 64$ frequency values, $f = k/N$, $k = 0 \ldots 63$,




```
X=fft(x,64)
plot([0:63]/64,1/16*abs(X).^2)
```

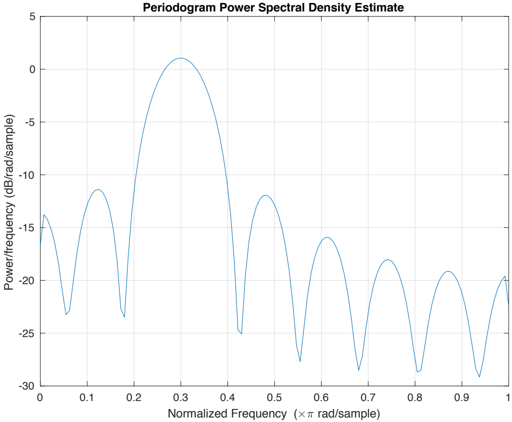
Example: A cosine

With more zero-padding, $N = 1024$, we see the true shape of the spectrum.




```
X=fft(x,1024)
plot([0:511]/1024,1/16*abs(X(1:512)).^2)
```

Example: A cosine



periodogram(x)