Stationary stochastic processes

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Lecture 2

Webpage: www.maths.lu.se/kurshemsida/fmsf10masc04/

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Yesterday news

Definition 2.1: For any stochastic process, we define a

- mean value function, $m(t) = E[X(t)]$,
- covariance function, $r(s, t) = C[X(s), X(t)]$,
- variance function, $v(t) = V[X(t)] = r(t, t)$,
- correlation function, $\rho(s, t) = \frac{C[X(s),X(t)]}{\sqrt{V[X(s)]V[X(t)]}}$.

Yesterday news

A weakly stationary stochastic process is defined with

- mean value function $m(t) = E[X(t)] = m$,
- covariance function $r(s,t) = r(t-s) = r(\tau)$,
- variance function $v(t) = r(t-t) = r(0)$,
- correlation function $\rho(\tau) = \frac{r(\tau)}{r(0)}$,

where $\tau = t - s$. (Definitions 2.4 and 2.5)

Additional properties (Theorem 2.2):

- $r(0) \geq 0$,
- $r(-\tau) = r(\tau)$,
- $r(0) \geq |r(\tau)|$.

Schedule for today

- Strictly stationary processes
- Correlation (covariance) function
- Estimation of mean value with examples
- Shortly, estimation of covariance function

Same covariance - different processes

A weakly stationary process is not unambiguously defined.

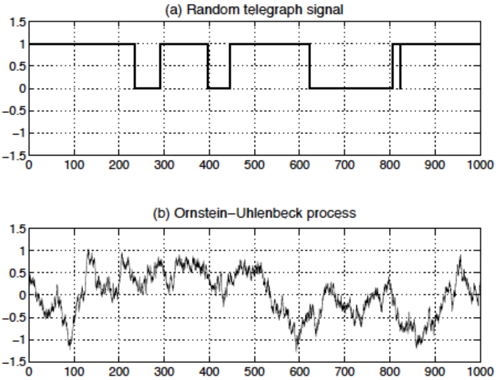


Figure 2.10 *Realizations of two processes with the same covariance function,* $r(\tau) = \sigma^2 e^{-\alpha|\tau|}$*: (a) random telegraph signal, (b) Ornstein-Uhlenbeck Gaussian process.*

Strictly stationary processes

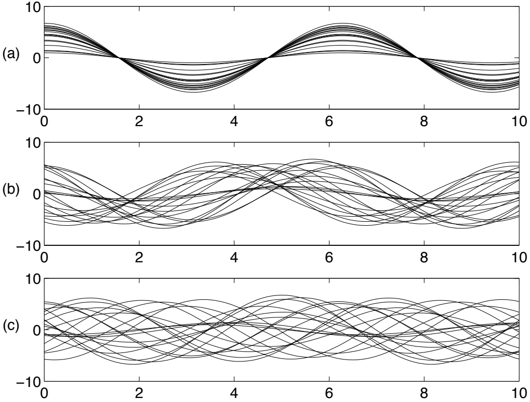
A process is strictly stationary if the distribution function remains unchanged after a shift of the time scale. A strictly stationary process is always weakly stationary.

In this course we deal with two strictly stationary processes:

- Random harmonic functions.
- Gaussian processes. (Lecture 3)

Random harmonic functions

Realizations of $X(t) = A\cos(t+\phi)$, where A is a stochastic variable and a) $\phi = 0$; b) $\phi \in U(0,\pi)$; c) $\phi \in U(0,2\pi)$.



The processes a) and b) are non-stationary and c) is strictly stationary, where the covariance function is $r(\tau) = \frac{E[A^2]}{2}\cos(\tau)$.

Correlation function

The correlation function for any stochastic process $X(t)$ is defined as

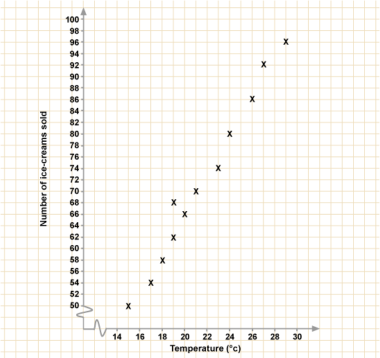
$$\rho(s,t) = \frac{C[X(s),X(t)]}{\sqrt{V[X(s)]V[X(t)]}},$$

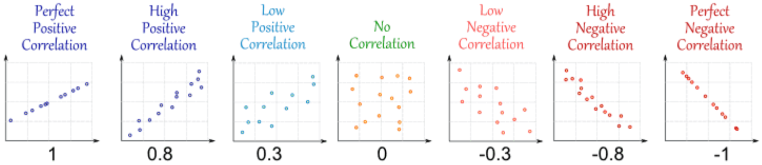
which is simplified to

$$\rho(\tau) = \frac{r(\tau)}{\sqrt{r(0)r(0)}} = \frac{r(\tau)}{r(0)},$$

for a weakly stationary stochastic process where $\tau = t - s$. Therefore, the correlation function is just a normalized form of the covariance function, where $\rho(0) = 1$.

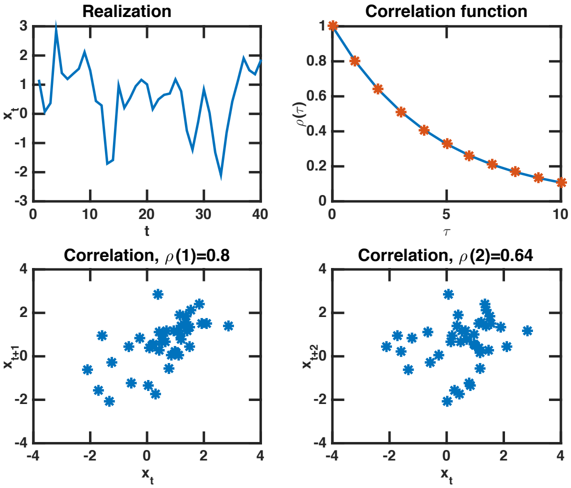
Correlation example





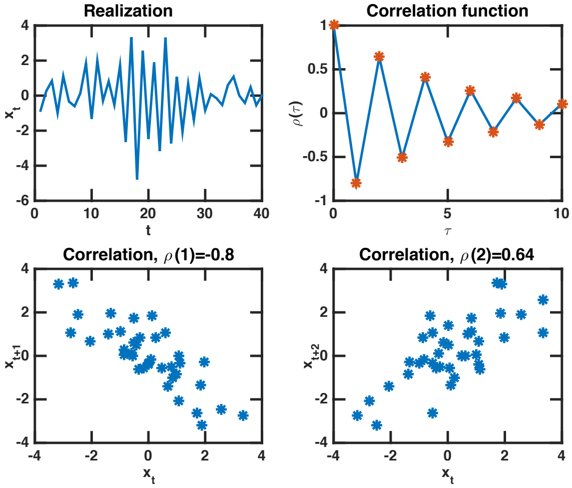
Example 1

The correlation function $\rho(\tau) = (0.8)^{|\tau|}$, $\tau = 0, \pm1, \pm2, \ldots$ for the weakly stationary stochastic process X_t, $t = 0, \pm1, \pm2, \ldots$.



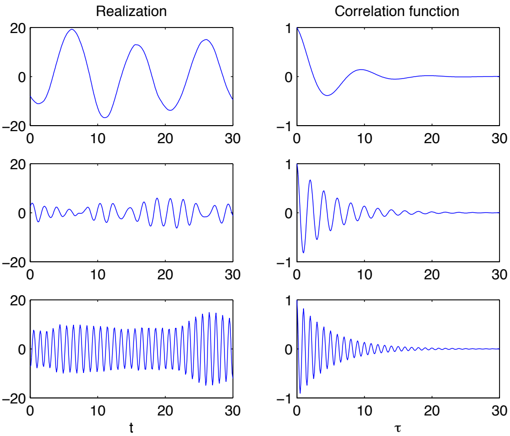
Example 2

The correlation function $\rho(\tau) = (-0.8)^{|\tau|}$, $\tau = 0, \pm 1, \pm 2, \ldots$ for the weakly stationary stochastic process X_t, $t = 0, \pm 1, \pm 2, \ldots$.



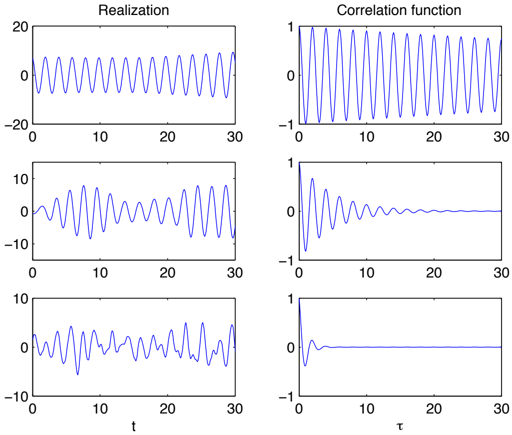
Data and correlation function

The main period of data is reflected in the period of the correlation and covariance function.



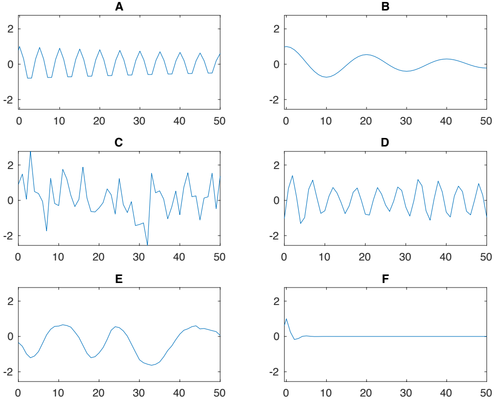
Data and correlation function

High correlation for large values of τ is connected to a more periodic data sequence.



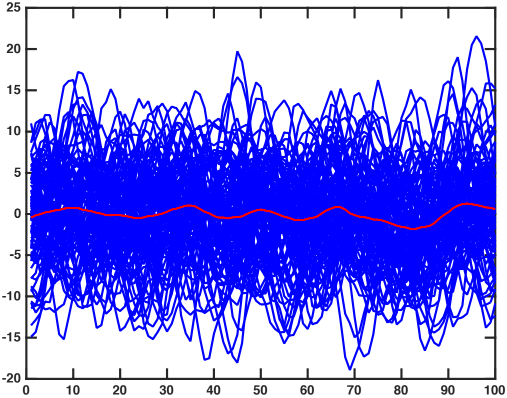
Exercise

Realizations from three different stationary processes and corresponding covariance functions are presented. Identify the realizations and covariance functions respectively. Finally combine the figures corresponding to each process.



Ensemble mean

For a weakly stationary stochastic process the mean value function, $m(t) = m$. When the number of ensembles approaches infinity, the mean values for all t approach m.



Mean value over time

As we usually have just a one or a few ensemble(s), the averaged value of just one realization of data x_t, $t = 1 \ldots n$, using the mean value over time

$$\hat{m}_n = \frac{1}{n} \sum_{t=1}^{n} X_t,$$

gives an unbiased (väntevärdesriktigt) estimate of m, as

$$E[\hat{m}_n] = \frac{1}{n} \sum_{t=1}^{n} E[X_t] = \frac{1}{n} \underbrace{(m + m + \ldots + m)}_{n} = m.$$

Linearly ergodic

A stationary process is linearly ergodic, (linjärt ergodisk), as the ensemble mean can be estimated using the mean value over time.

Old exam exercise (modified)

The average level m of a stationary stochastic process, $Y_t = m + X_t$, $t = 0, \pm 1, \pm 2, \ldots$, should be estimated. A model of the process X_t is defined by

$$X_t = e_t - 2e_{t-1} + e_{t-2},$$

where e_t, $t = 0, \pm 1, \pm 2 \ldots$, is white noise with expected value zero and variance one. One can choose between two estimates for m,

$$\hat{m}_1 = \frac{Y_t + Y_{t-1}}{2}$$

or

$$\hat{m}_2 = \frac{Y_t + Y_{t-2}}{2}.$$

Which is the most optimal estimator, $\hat{m}_1$ or $\hat{m}_2$?

Variance of $\hat{m}_n$

The variance is calculated as

$$V[\hat{m}_n] = C[\frac{1}{n}\sum_{t=1}^{n} X_t, \frac{1}{n}\sum_{s=1}^{n} X_s] = \frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n} r(s-t).$$

With $s - t = u$ we get

$$V[\hat{m}_n] = \frac{1}{n^2}\sum_{u=-n+1}^{n-1}(n-|u|)r(u).$$

For large n,

$$V[\hat{m}_n] \approx \frac{1}{n}\sum_{u} r(u).$$

If $V[\hat{m}_n] \to 0$ when $n \to \infty$, $\hat{m}_n$ is consistent, (konsistent).

Exercise 2.16: Numbers in the average

Suppose that, $X_t, t = 0, \pm1, \pm2...$, is a stationary process with unknown mean m, known variance σ^2 and correlation function

$$\rho(\tau) = 0.5^{|\tau|}, \quad \tau = 0, \pm1, \pm2, \ldots.$$

We would like to estimate m by averaging N_1 consecutive samples of the process. Suppose that N_1 is large and approximate the variance of the estimator.

Also, find the value of N_2, that would have been necessary in order to achieve the same variance, if the elements of the process had been uncorrelated.

Estimation of the covariance function

A weakly stationary process is ergodic of second order if the covariance function fulfills $\frac{1}{n}\sum_{\tau=1}^{n} r(\tau)^2 \to 0$ when $n \to \infty$. We can estimate the covariance function as the time average from one realization as,

$$\hat{r}_n(\tau) = \frac{1}{n}\sum_{t=1}^{n-\tau}(X_t - m)(X_{t+\tau} - m).$$

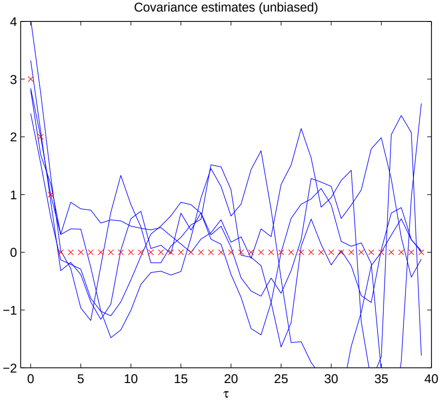
If m is unknown the estimate $\hat{m}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$ is used. The estimate $\hat{r}_n(\tau)$ is biased as

$$E[\hat{r}_n(\tau)] = \frac{1}{n}\sum_{t=1}^{n-\tau} r(\tau) = \frac{1}{n}(n-\tau)r(\tau).$$

When $n \to \infty$, $E[\hat{r}_n(\tau)] \to r(\tau)$, i.e. the estimate is asymptotically (asymptotiskt) unbiased. (Theorem 2.5)

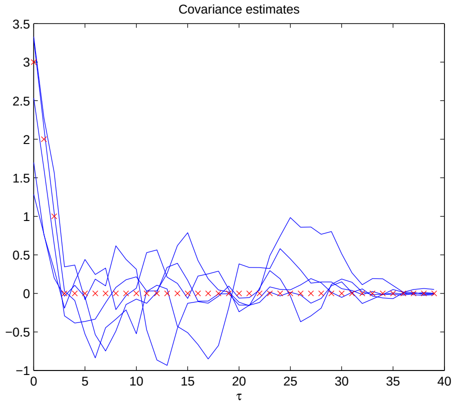
Example: Unbiased estimate

The covariance estimate is divided with $n - \tau$. We see that the variance of the estimates for large τ will be large as they are based on very few data values. The true covariance function is shown with crosses.



Example: Biased estimate

To suppress the variance for large τ the biased covariance estimate is used, dividing by n.



Therefore the biased covariance estimate is most often applied.