Stationary stochastic processes

Maria Sandsten

Lecture 2

September 4 2018

Yesterday news

Definition 2.1: For any stochastic process, we define a

- mean value function, $m(t) = E[X(t)]$,
- covariance function, $r(s, t) = C[X(s), X(t)]$,
- variance function, $v(t) = V[X(t)] = r(t, t)$,
- correlation function, $\rho(s, t) = \frac{C[X(s),X(t)]}{\sqrt{V[X(s)]V[X(t)]}}$.

Yesterday news

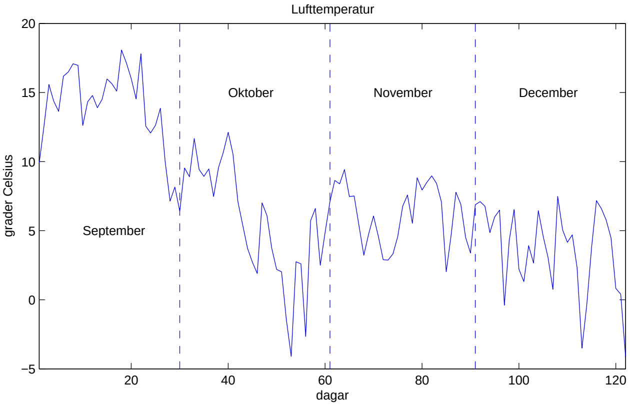
Definitions 2.4 and 2.5:

A weakly stationary stochastic process is defined with

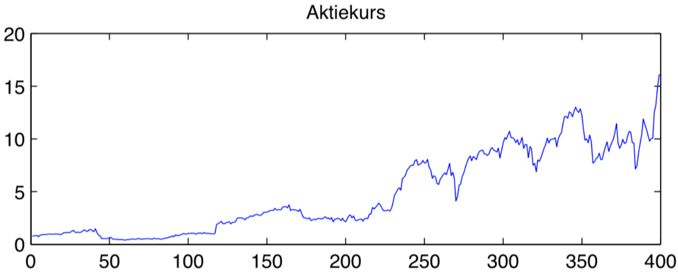
- mean value function $m(t) = E[X(t)] = m$,
- covariance function $r(s,t) = r(t-s) = r(\tau)$,
- variance function $v(t) = r(t-t) = r(0)$,
- correlation function $\rho(\tau) = \frac{r(\tau)}{\sqrt{r(0)r(0)}} = \frac{r(\tau)}{r(0)}$,

where $\tau = t - s$,

Temperature data



Financial data



days

More properties of a weakly stationary process

For a real-valued weakly stationary stochastic process we have (see Theorem 2.2):

- non-negative variance

$$r(0) \geq 0,$$

- symmetrical covariance function

$$r(-\tau) = r(\tau),$$

- variance is the maximum of the covariance function

$$r(0) \geq |r(\tau)|.$$

(Complex-valued processes are not covered in this course.)

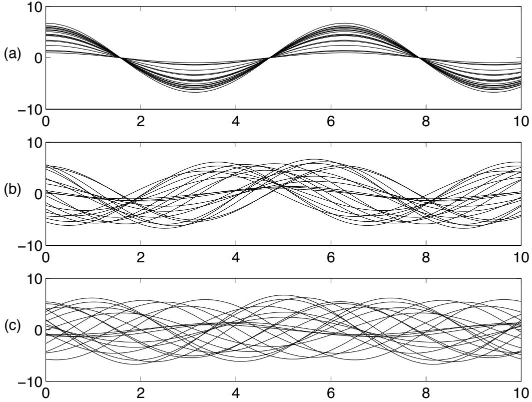
Old exam problem (modified)

Determine which of the following that correspond to a valid covariance function of a weakly discrete-time real-valued stationary stochastic process. Justify your answers.

- a) $r(\tau) = \begin{cases} 1.25 & \tau = 0 \\ 0.5 & \tau = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$
- b) $r(\tau) = \sin(\tau)$,
- c) $r(\tau) = \begin{cases} 0 & \tau = 0 \\ 1 & \tau = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$
- d) $r(\tau) = \cos(\tau)$.

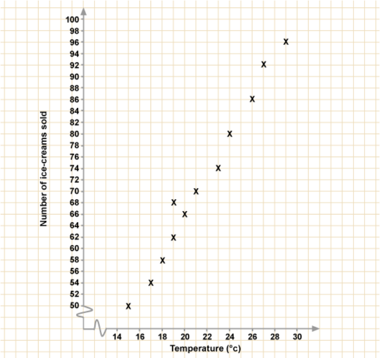
Example of a strictly stationary process

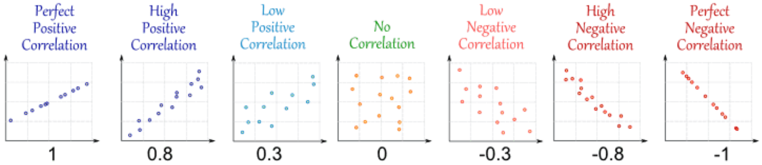
Realizations of $X(t) = A\cos(t+\phi)$, where A is a stochastic variable and a) $\phi = 0$; b) $\phi \in U(0,\pi)$; c) $\phi \in U(0,2\pi)$.



The processes a) and b) are non-stationary and c) is strictly stationary, where the covariance function is $r(\tau) = \frac{E[A^2]}{2}\cos(\tau)$.

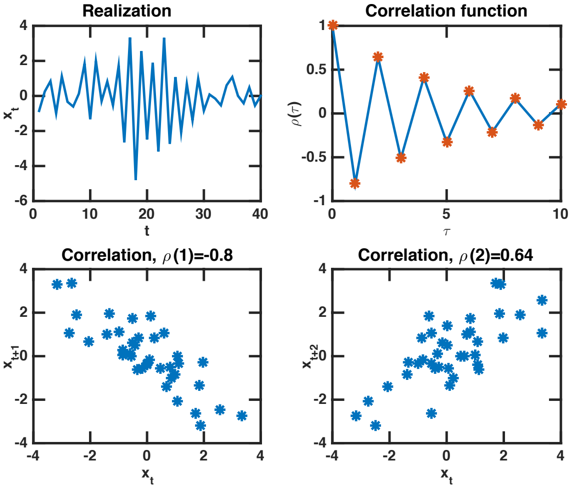
Correlation example





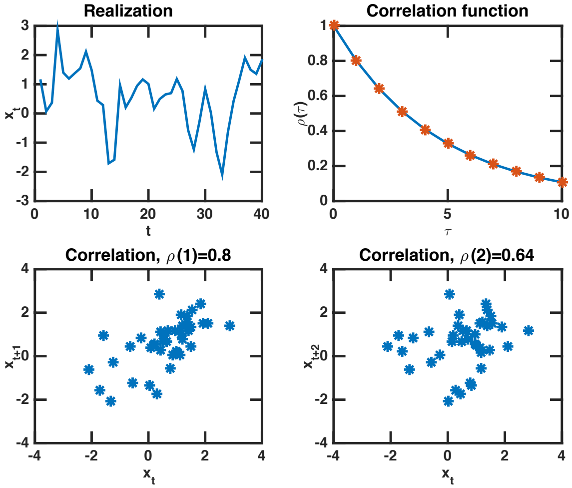
Example 1

Correlation function $\rho(\tau) = (-0.8)^{|\tau|}$.



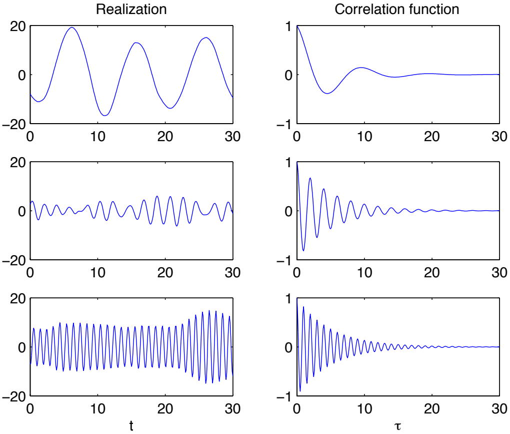
Example 2

Correlation function $\rho(\tau) = 0.8^{|\tau|}$.



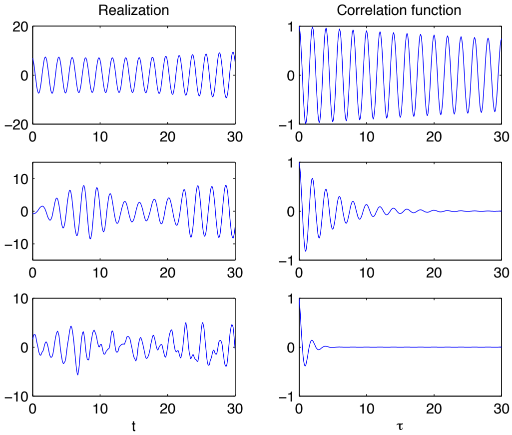
Data and correlation function

The main period of data is reflected in the period of the correlation and covariance function.



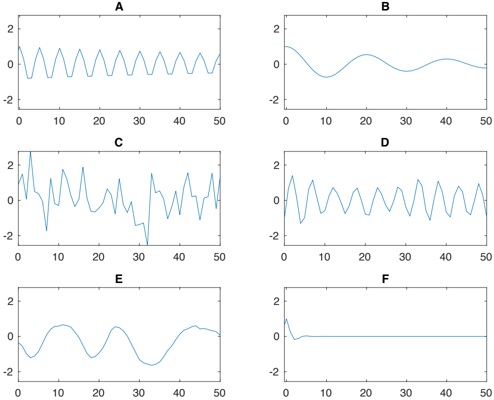
Data and correlation function

High correlation for large values of τ is connected to a more periodic data sequence.



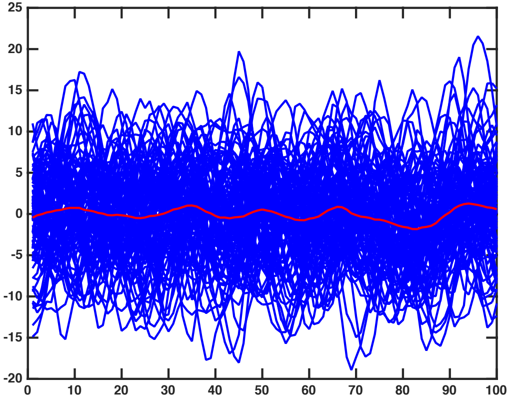
Exercise

Realizations from three different stationary processes and corresponding covariance functions are presented. Identify the realizations and covariance functions respectively. Finally combine the figures corresponding to each process.



Estimation of mean value

When the number of ensembles - realizations, approaches infinity, the mean values for all t approach m for a stationary stochastic process.



Estimation of mean value

A stationary process is linearly ergodic, as the ensemble mean can be estimated using the mean value over time,

$$\hat{m}_n = \frac{1}{n} \sum_{t=1}^{n} X_t,$$

which gives an unbiased estimate of m, as

$$E[\hat{m}_n] = \frac{1}{n} \sum_{t=1}^{n} E[X_t] = \frac{1}{n} \underbrace{(m + m + \ldots + m)}_{n} = m.$$

Variance of $\hat{m}_n$

The variance is calculated as

$$V[\hat{m}_n] = C[\frac{1}{n}\sum_{t=1}^{n} X_t, \frac{1}{n}\sum_{s=1}^{n} X_s] = \frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n} r(s-t).$$

With $s - t = u$ we get

$$V[\hat{m}_n] = \frac{1}{n^2}\sum_{u=-n+1}^{n-1}(n-|u|)r(u).$$

For large n,

$$V[\hat{m}_n] \approx \frac{1}{n}\sum_{u} r(u).$$

If $V[\hat{m}_n] \to 0$ when $n \to \infty$, $\hat{m}_n$ is consistent, (konsistent).

Old exam exercise (modified)

The average level m of a stationary stochastic process, $Y_t = m + X_t,\ t = 0, \pm 1, \pm 2, \ldots$, should be estimated. A model of the process X_t is defined by

$$X_t = e_t - 2e_{t-1} + e_{t-2},$$

where $e_t,\ t = 0, \pm 1, \pm 2 \ldots$, is white noise with expected value zero and variance one. One can choose between two estimates for m,

$$\hat{m}_1 = \frac{Y_t + Y_{t-1}}{2}$$

or

$$\hat{m}_2 = \frac{Y_t + Y_{t-2}}{2}.$$

Which is the most optimal estimator, $\hat{m}_1$ or $\hat{m}_2$?

Exercise 2.16: Number in the average

Suppose that, $\{X_t, t = 0, \pm1, \pm2...\}$, is a stationary process with unknown mean m, known variance σ^2 and correlation function

$$\rho(\tau) = 0.5^{|\tau|}, \quad \tau = 0, \pm1, \pm2, \ldots.$$

We would like to estimate m by averaging N_1 consecutive samples of the process. Suppose that N_1 is large and approximate the variance of the estimator.

Also, find the value of N_2, that would have been necessary in order to achieve the same variance, if the elements of the process had been uncorrelated.

Estimation of covariance function

If X_t, $t = 1, 2 \ldots$ is a stationary Gaussian process and the covariance function fulfills

$$\frac{1}{n} \sum_{\tau=1}^{n} r(\tau)^2 \to 0 \text{ when } n \to \infty$$

then the process is ergodic of second order and we can estimate expected value and the covariance function as time averages from one realization.

Estimation of covariance function

Theorem 2.5: The covariance function is estimated as

$$\hat{r}_n(\tau) = \frac{1}{n}\sum_{t=1}^{n-\tau}(X_t - m)(X_{t+\tau} - m),$$

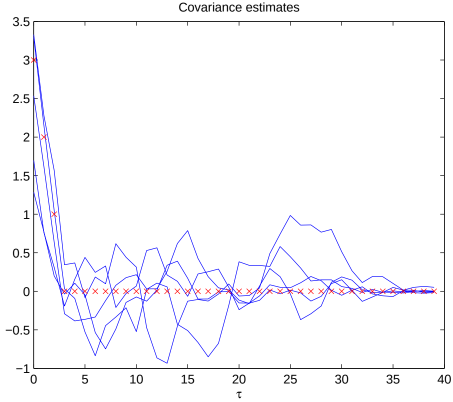
which is a biased estimate as

$$E[\hat{r}_n(\tau)] = \frac{1}{n}\sum_{t=1}^{n-\tau} r(\tau) = \frac{1}{n}(n-\tau)r(\tau).$$

When $n \to \infty$, $E[\hat{r}_n(\tau)] \to r(\tau)$, i.e. the estimate is asymptotically (asymptotiskt) unbiased. If m is unknown the estimate $\hat{m}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$ is used.

Example: Biased estimate

The usual (biased) covariance estimates from a few realizations of a stationary stochastic process are made. The true covariance function is shown with crosses.



Example: Unbiased estimate

The covariance estimate is divided with $n - \tau$ instead of n (unbiased). The variation of the estimates for large τ will be high as they are based on very few data values.