Stationary stochastic processes

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Lecture 11

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Examination information

Stationary Stochastic Processes

News HT 2019

The written exam is given Friday November 1 8.00-13.00. The locations will be divided between Victoriastadion, Lovisastigen 2-4 and the Centre of Mathematical Sciences-building, Sölvegatan 18. Your location is decided according to your anonymouscode,

MASC04: go to Vic 3A-C.

FMSF10: F10 001xxx-090xxx go to Vic 3A-C.

FMSF10: F10 091xxx-250xxx and others go to MH:362 A-D (3d floor).

The exam will be in english. You are allowed to bring a pocket calculator and the table of formulas of this course. You can also bring some small 'high school' table of formulas in mathematics (e.g. TEFYMA). Table of formulas must not contain any marks or comments.

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Important course topics

- Properties of a stationary stochastic process.
- Calculation of covariance functions from spectral densities.
- Characteristics of covariance function, spectral density and corresponding realization.
- Calculation of mean, covariance functions and spectral densities of filtered processes.
- Calculation of parameters, covariance functions and spectral densities of AR-process and MA-processes (pole-zero plots).
- Cross-covariance and cross-spectrum calculations.
- Calculations of derivatives and integrals.
- Optimal filters (MSE, Wiener and matched filter).
- Sampling.
- Variance calculation of the mean value estimate.
- Knowledge of basic spectral estimation techniques.

A stationary stochastic process

A continuous time process, $X(t)$, $t \in \mathbb{R}$, or a discrete time process $X_t, t = 0, \pm 1, \ldots$, is weakly stationary if,

- The expected value, $E[X(t)] = m$
- The covariance function, $C[X(s), X(t)] = r(t-s) = r(\tau)$
- The variance, $V[X(t)] = r(0)$
- The correlation function, $\rho(\tau) = r(\tau)/r(0)$

For a real-valued stationary stochastic process we have

- $V[X(t)] = r(0) \geq 0$
- $r(-\tau) = r(\tau)$
- $|r(\tau)| \leq r(0)$

Spectral density

For a weakly stationary process there exists a, positive, symmetrical and integrable spectral density function $R(f)$ such that,

$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f\tau} df, \quad R(f) = \int_{-\infty}^{\infty} r(\tau) e^{-i2\pi f\tau} d\tau,$$

in continuous time, and

$$r(\tau) = \int_{-1/2}^{1/2} R(f) e^{i2\pi f\tau} df, \quad R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f\tau},$$

in discrete time. If $E[X(t)] = 0$ the power and the variance is

$$E[X^2(t)] = V[X(t)] = r(0) = \int_{-\infty}^{\infty} R(f) df.$$

Discrete time white noise

For discrete time white noise,

$$R(f) = \sigma^2, \quad -1/2 < f \leq 1/2,$$

and

$$r(\tau) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = \pm 1, \pm 2, \ldots \end{cases}$$

Random harmonic function

The random harmonic function

$$X(t) = A\cos(2\pi f_0 t + \phi), \quad t \in \mathbb{R},$$

is a stationary process with $A > 0$ and $\phi \in Rect(0, 2\pi)$. The covariance function is

$$r(\tau) = \frac{E[A^2]}{2}\cos(2\pi f_0 \tau), \;\; \tau \in \mathbb{R},$$

for all $f_0 > 0$ and the spectral density is defined as

$$R(f) = \frac{E[A^2]}{4}(\delta(f - f_0) + \delta(f + f_0)).$$

The random harmonic function is also defined for discrete time with $0 < f_0 \leq 0.5$

Filtering of stationary processes

For continuous time processes the output $Y(t)$, $t \in \mathbb{R}$, is obtained from the input $X(t)$, $t \in \mathbb{R}$, through

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du,$$

where $h(t)$ is the impulse response. The corresponding frequency function is

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt.$$

For discrete time processes, the integrals are changed to sums.

Filtering of stationary processes

The mean value,

$$m_Y = m_X \int_{-\infty}^{\infty} h(u)du = m_X H(0),$$

the covariance function,

$$r_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u)h(v)r_X(\tau + u - v)dudv,$$

and the spectral density,

$$R_Y(f) = |H(f)|^2 R_X(f),$$

For discrete time processes, the integrals are changed to sums.

The MA(q)-process

A moving average process of order q, MA(q), is given by

$$X_t = c_0 e_t + c_1 e_{t-1} + \ldots + c_q e_{t-q},$$

where e_t, $t = 0, \pm 1, \pm 2, \ldots,$ is a zero-mean white noise Gaussian noise with variance σ^2. The expected value $m_X = 0$ and the covariance function is

$$r_X(\tau) = C[X_t, X_{t+\tau}] \neq 0 \quad |\tau| \leq q.$$

The spectral density is the discrete time Fourier transform of $r_X(\tau)$ or

$$R_X(f) = \sigma^2 |c_0 + c_1 e^{-i2\pi f} + \ldots + c_q e^{-i2\pi f q}|^2.$$

The AR(p)-process

An auto-regressive process of order p, AR(p), is given by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t,$$

where, e_t, $t = 0, \pm 1, \pm 2 \ldots$, is zero-mean white Gaussian noise with variance σ^2. The expected value, $m_X = 0$, and the covariance function solves the Yule-Walker equations

$$r_X(\tau) + a_1 r_X(\tau - 1) + \ldots + a_p r_X(\tau - p) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = 1, 2, \ldots \end{cases}$$

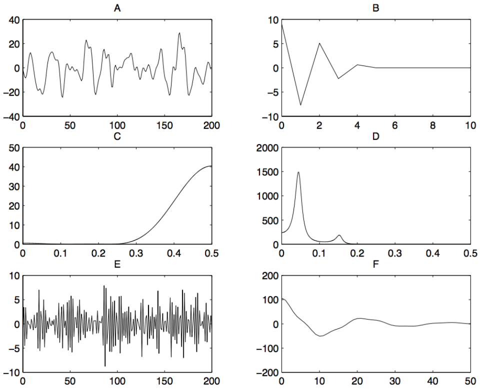
The spectral density is

$$R_X(f) = \frac{\sigma^2}{|1 + a_1 e^{-i2\pi f} + \ldots + a_p e^{-i2\pi f p}|^2}.$$

Old exam 180108-1

The following figures show realizations, covariance functions and spectral densities of one AR(p)-process and one MA(q)-process. Determine, with a motivation, what figures that are realizations, covariance functions and spectral densities. Also state which realization, covariance function and spectral density that are connected. Decide of which type the different processes are (AR or MA) and which orders they have. The order can be assumed to be less than 10 for both processes. Important! To receive full number of credits correct motivations have to be given for all answers.

(10p)



Old exam 080306-5

A weakly stationary process $X_t,\ t = 0, \pm 1, \pm 2 \ldots$, is defined as

$$X_t - X_{t-1} + 0.5X_{t-2} = e_t + B$$

where e_t is white noise with expected value zero and variance $\sigma^2 = 1$. The process B is a stochastic variable with variance σ_B^2, which is independent of e_t. Compute the covariance function $r_X(\tau)$ for $\tau = 0, \pm 1, \pm 2, \pm 3$.

(20p)

Solution

Define a new process $Z_t = X_t - cB$ to be zero-mean and accordingly an AR(2)-process defined as $Z_t - Z_{t-1} + 0.5Z_{t-2} = e_t$. We apply $X_t = Z_t + cB$, in $X_t - X_{t-1} + 0.5X_{t-2} = e_t + B$ and get

$$E[Z_t + cB] - E[Z_{t-1} + cB] + 0.5E[Z_{t-2} + cB] = E[e_t] + E[B].$$

As $E[Z_t] = 0$ and $E[e_t] = 0$ by definition,

$$cE[B] - cE[B] + 0.5cE[B] = E[B],$$

giving $c = 2$. The Yule-Walker-equations for Z_t are the same as in the AR(2)-example from Example 8.pdf among the lecture notes, giving for $\sigma^2 = 1$, $r_Z(0) = 2.4$, $r_Z(\pm 1) = 1.6$, $r_Z(\pm 2) = 0.4$ and $r_Z(\pm 3) = -0.4$. Accordingly

$$r_X(\tau) = C[Z_t + 2B, Z_{t+\tau} + 2B] = r_Z(\tau) + 4\sigma_B^2.$$

(Also read Example 4.12, page 105).

Cross-covariance, cross spectrum and coherence spectrum

The cross-covariance function is defined as

$$r_{X,Y}(\tau) = C[X(t), Y(t+\tau)].$$

Note that $r_{X,Y}(\tau) \neq r_{X,Y}(-\tau)$ but $r_{X,Y}(\tau) = r_{Y,X}(-\tau)$.
The complex-valued cross spectrum $R_{X,Y}(f)$ is defined so that

$$r_{X,Y}(\tau) = \int R_{X,Y}(f) e^{i2\pi f \tau} df.$$

The quadratic coherence spectrum is defined

$$\kappa^2_{X,Y}(f) = \frac{|R_{X,Y}(f)|^2}{R_X(f) R_Y(f)}, \quad 0 \leq \kappa^2_{X,Y} \leq 1.$$

The same formulations apply for discrete time processes.

Differentiation

A weakly stationary process $X(t)$, $t \in \mathbb{R}$, is said to be differentiable in quadratic mean with the derivative $X'(t)$ if $r_X(\tau)$ is twice differentiable or if

$$V[X'(t)] = \int_{-\infty}^{\infty} (2\pi f)^2 R_X(f) df < \infty.$$

We have $m_{X'} = 0$,

$$r_{X'}(\tau) = -r_X''(\tau),$$

and

$$R_{X'}(f) = (2\pi f)^2 R_X(f).$$

The cross-covariance function, $r_{X,X'}(t, t+\tau) = r_X'(\tau)$ and $r_{X,X'}(t,t) = 0$.

Old exam 171028-3

A zero-mean, weakly stationary, Gaussian process $X(t)$, $t \in \mathbb{R}$ has the spectral density

$$R_X(f) = \pi e^{-2\pi|f|}.$$

State, with a short motivation, for each of the following statements if it is right or wrong. Each correct answer with motivation gives 2 credits.

a) The covariance function is $r_X(\tau) = \frac{1}{1+\tau^2}$.

b) The process is differentiable in quadratic mean.

c) The variance $V[(X(0) + X(t))/2]$ approaches zero 0 when $t \to \infty$.

d) After filtering through a linear filter with frequency function $H(f) = 1 + if$, the spectral density of the process is constant for all frequencies.

e) A new process $Y(t) = 3X(t) - 2X'(t)$ is created. The process $Y(t)$ is a Gaussian process.

(10p)

Old exam 181102-2

A stationary Gaussian process $X(t)$, $t \in \mathbb{R}$, has expected value $E[X(t)] = 2$ and covariance function

$$r_X(\tau) = e^{-\tau^2}.$$

Compute the probability,

$$P(X'(t) \geq X(t)).$$

(10p)

Optimal filters

- Mean square error optimal filter, minimize $E[(Y(t) - S(t))^2]$.
- The matched filter for binary detection is

$$h_{opt}(u) = s(T - u),$$

for the zero-mean white noise disturbance case. For equal decision errors, the decision level $k = s_{out}(T)/2$ with errors $\alpha = \beta = 1 - \Phi(\frac{s_{out}(T)}{2\sigma_N})$.

- The Wiener filter frequency function is

$$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)},$$

where $R_S(f)$ and $R_N(f)$ are the zero-mean signal and noise spectral densities respectively.

Sampling

The continuous time process $Y(t)$, $t \in \mathbb{R}$ is sampled to the discrete time sequence $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$. The covariance function is

$$r_Z(\tau) = r_Y(\tau), \;\; \tau = 0, \pm d, \pm 2d, \ldots$$

and the spectral density

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) \;\; -f_s/2 < f \leq f_s/2.$$

with $f_s = 1/d$ as the sampling frequency. With $\tau = nd$, $r_Z(\tau)$ is converted to $r_X(n)$, $n = 0, \pm 1, \pm 2, \ldots$, and

$$R_X(\nu) = f_s \cdot R_Z(\nu f_s),$$

for $\nu = f \cdot d = f/f_s$.

Estimation of mean

If $X(t)$, $t = 1, 2 \ldots$ is weakly stationary with the unknown expected value m then

$$\hat{m}_n = \frac{1}{n}\sum_{t=1}^{n} X(t)$$

is an unbiased estimate of m as $E[\hat{m}_n] = m$. The variance is

$$V[\hat{m}_n] = C[\frac{1}{n}\sum_{t=1}^{n} x(t), \frac{1}{n}\sum_{s=1}^{n} x(s)] = \frac{1}{n^2}\sum_{\tau=-n+1}^{n-1} (n - |\tau|)r(\tau).$$

For large n,

$$V[\hat{m}_n] \approx \frac{1}{n}\sum_{\tau} r(\tau).$$

Old exam 171028-5

We define the weakly stationary Gaussian process $Y(t) = m + X(t)$, $t \in \mathbb{R}$, where m is an unknown constant expected value. The process $X(t)$ is assumed to have $E[X(t)] = 0$ and spectral density

$$R_X(f) = \begin{cases} 1 & \text{for} \quad |f - 2.1| \leq 0.1, \\ 1 & \text{for} \quad |f + 2.1| \leq 0.1 \\ 0 & \text{otherwise.} \end{cases}$$

An estimate of m should be found as

$$\widehat{m}_n = \frac{Y(d) + Y(2d) + \dots Y(nd)}{n},$$

by sampling the process $Y(t)$ with sample distance d. There are two possible choices of sample distances, $d = 1/4$ and $d = 1/2$.

a) Calculate the covariance function $r_Y(\tau)$.

(4p)

b) Determine the spectral density of the sampled process $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \dots$ for the two suggested sample distances, $d = 1/4$ and $d = 1/2$.

(10p)

c) Determine $nV[\widehat{m}_n]$, when $n \to \infty$ for $d = 1/4$ and $d = 1/2$. Which of the suggested sample distances would you prefer?

(6p)

Spectrum estimation

The periodogram is defined as

$$\widehat{R}_x(f) = \frac{1}{n} \left| \sum_t x(t)w(t)e^{-i2\pi ft} \right|^2,$$

where $w(t)$ is a data window. With the Hanning window, the spectrum estimate will have better leakage properties (lower sidelobes), although the resolution is somewhat degraded, (wider mainlobe), in comparison to the rectangular window.

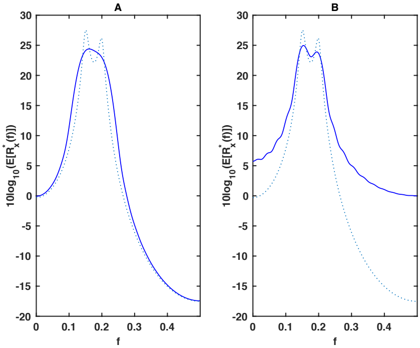
The variance of the periodogram is

$$V[\widehat{R}_x(f)] \approx R_X^2(f) \;\; 0 < |f| < 0.5$$

Dividing the sequence into K possibly overlapping sequences and calculating the average of K approximately uncorrelated estimates (Welch method), reduces the variance to

$$V[\widehat{R}_{mv}(f)] \approx \frac{1}{K} R_X^2(f).$$

Old exam 180108-3



a) In the figures above, the expected values of the periodogram the rectangular and the Hanning window, calculated from n samples, are illustrated with solid lines. Which of the figures shows the expected value using the two windows. Motivate your answer.

b) The sequence is divided into K sequences and the final estimate is calculated as the average of K uncorrelated periodograms. How much does the variance decrease in comparison to the periodogram in a)?

(10p)