Stationary stochastic processes

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Lecture 11

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Important topics

- Properties of a stationary stochastic process.
- Calculation of covariance functions from spectral densities.
- Characteristics of covariance function, spectral density and corresponding realization.
- Calculation of mean, covariance functions and spectral densities of filtered processes.
- Calculation of parameters, covariance functions and spectral densities of AR-process and MA-processes (pole-zero plots).
- Cross-covariance and cross-spectrum calculations.
- Calculations of derivatives and integrals.
- Optimal filters (MSE, Wiener and matched filter).
- Sampling.
- Variance calculation of the mean value estimate.
- Knowledge of basic spectral estimation techniques.

Examination information

The course is concluded with a written exam, Friday November 2, 14.00-19.00, at Victoriastadion, Vic:2 and 3A.

The exam will be in english. You are allowed to bring a pocket calculator and the table of formulas of this course. You can also bring some small 'high school' table of formulas in mathematics (e.g. TEFYMA). Table of formulas must not contain any marks or comments.

Old exams

Old exams

exam20161222 - solution20161222

exam20170817 - solution20170817

Additional Question Times for the Exam

Wed Oct 24 10.15-12.00 MH:227.

Tue Oct 30 10.15-12.00 MH:227.

Thu Nov 1 10.15-12.00 MH:227.

A stationary stochastic process

A continuous time process, $X(t)$, $t \in \mathbb{R}$, or a discrete time process X_t,$t = 0, \pm 1, \ldots$, is weakly stationary if,

- The expected value, $E[X(t)] = m$
- The covariance function, $C[X(s), X(t)] = r(t - s) = r(\tau)$
- The variance, $V[X(t)] = r(0)$
- The correlation function, $\rho(\tau) = r(\tau)/r(0)$

For a real-valued stationary stochastic process we have

- $V[X(t)] = r(0) \geq 0$
- $r(-\tau) = r(\tau)$
- $|r(\tau)| \leq r(0)$

Spectral density

For a weakly stationary process there exists a, positive, symmetrical and integrable spectral density function $R(f)$ such that,

$$r(\tau) = \int_{-\infty}^{\infty} R(f)e^{i2\pi f\tau}df, \quad R(f) = \int_{-\infty}^{\infty} r(\tau)e^{-i2\pi f\tau}d\tau,$$

in continuous time, and

$$r(\tau) = \int_{-1/2}^{1/2} R(f)e^{i2\pi f\tau}df, \quad R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau)e^{-i2\pi f\tau},$$

in discrete time. If $E[X(t)] = 0$ the power and the variance is

$$E[X^2(t)] = V[X(t)] = r(0) = \int_{-\infty}^{\infty} R(f)df.$$

Discrete time white noise

For discrete time white noise, $R(f) = \sigma^2$, $-1/2 < f \leq 1/2$, and

$$r(\tau) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = \pm 1, \pm 2, \ldots \end{cases}$$

For continuous time white noise, $R(f) = \sigma^2$, $-\infty < f < \infty$, and

$$r(\tau) = \sigma^2 \delta(\tau), \quad \tau \in \mathbb{R},$$

which we use in calculations although it does not exist in reality.

Random harmonic function

The random harmonic function

$$X(t) = A\cos(2\pi f_0 t + \phi), \quad t \in \mathbb{R},$$

is a stationary process with $A > 0$ and $\phi \in Rect(0, 2\pi)$. The covariance function is

$$r(\tau) = \frac{E[A^2]}{2}\cos(2\pi f_0 \tau), \;\; \tau \in \mathbb{R},$$

for all $f_0 > 0$ and the spectral density is defined as

$$R(f) = \frac{E[A^2]}{4}(\delta(f - f_0) + \delta(f + f_0)).$$

The random harmonic function is also defined for discrete time with $0 < f_0 \leq 0.5$

Filtering of stationary processes

For continuous time processes the output $Y(t)$, $t \in \mathbb{R}$, is obtained from the input $X(t)$, $t \in \mathbb{R}$, through

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du,$$

where $h(t)$ is the impulse response. The corresponding frequency function is

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt.$$

For discrete time processes, the integrals are changed to sums.

Filtering of stationary processes

The mean value,

$$m_Y = m_X \int_{-\infty}^{\infty} h(u)du = m_X H(0),$$

the covariance function,

$$r_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u)h(v)r_X(\tau + u - v)dudv,$$

and the spectral density,

$$R_Y(f) = |H(f)|^2 R_X(f),$$

For discrete time processes, the integrals are changed to sums.

The MA(q)-process

A moving average process of order q, MA(q), is given by

$$X_t = c_0 e_t + c_1 e_{t-1} + \ldots + c_q e_{t-q},$$

where $e_t,\ t = 0, \pm 1, \pm 2, \ldots,$ is a zero-mean white noise Gaussian noise with variance σ^2. The expected value $m_X = 0$ and the covariance function is

$$r_X(\tau) = C[X_t, X_{t+\tau}] \neq 0 \quad |\tau| \leq q.$$

The spectral density is the discrete time Fourier transform of $r_X(\tau)$ or

$$R_X(f) = \sigma^2 |c_0 + c_1 e^{-i2\pi f} + \ldots + c_q e^{-i2\pi f q}|^2.$$

The AR(p)-process

An auto-regressive process of order p, AR(p), is given by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t,$$

where, e_t, $t = 0, \pm 1, \pm 2 \ldots$, is zero-mean white Gaussian noise with variance σ^2. The expected value, $m_X = 0$, and the covariance function solves the Yule-Walker equations

$$r_X(\tau) + a_1 r_X(\tau - 1) + \ldots + a_p r_X(\tau - p) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = 1, 2, \ldots \end{cases}$$

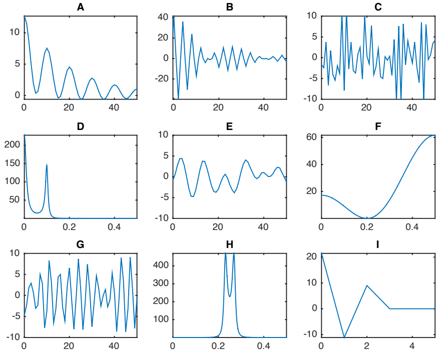
The spectral density is

$$R_X(f) = \frac{\sigma^2}{|1 + a_1 e^{-i2\pi f} + \ldots + a_p e^{-i2\pi f p}|^2}.$$

Exam 161222-1

The following figures are connected to three different processes. Determine, with a motivation, what figures that are realizations, covariance functions and spectral densities and specify which are connected to the same process. Decide and motivate of which type the different processes are (AR or MA) and which orders, p or q they have. The orders can be assumed to be smaller than 5.

(10p)



Exam 080306-5

A weakly stationary process X_t, $t = 0, \pm 1, \pm 2 \ldots$, is defined as

$$X_t - X_{t-1} + 0.5X_{t-2} = e_t + B$$

where e_t is white noise with expected value zero and variance $\sigma^2 = 1$. The process B is a stochastic variable with variance σ_B^2, which is independent of e_t. Compute the covariance function $r_X(\tau)$ for $\tau = 0, \pm 1, \pm 2, \pm 3$.

Solution

Define a new process $Z_t = X_t - cB$ to be zero-mean and accordingly an AR(2)-process defined as $Z_t - Z_{t-1} + 0.5Z_{t-2} = e_t$. We apply $X_t = Z_t + cB$, in $X_t - X_{t-1} + 0.5X_{t-2} = e_t + B$ and get

$$E[Z_t + cB] - E[Z_{t-1} + cB] + 0.5E[Z_{t-2} + cB] = E[e_t] + E[B].$$

As $E[Z_t] = 0$ and $E[e_t] = 0$ by definition,

$$cE[B] - cE[B] + 0.5cE[B] = E[B],$$

giving $c = 2$. The Yule-Walker-equations for Z_t are the same as in the AR(2)-example from lecture 9, $r_Z(0) = 2.4$, $r_Z(\pm 1) = 1.6$, $r_Z(\pm 2) = 0.4$ and $r_Z(\pm 3) = -0.4$. Accordingly

$$r_X(\tau) = C[Z_t + 2B, Z_{t+\tau} + 2B] = r_Z(\tau) + 4\sigma_B^2.$$

(Also read Example 4.12, page 105).

Cross-covariance, cross spectrum and coherence spectrum

The cross-covariance function is defined as

$$r_{X,Y}(\tau) = C[X(t), Y(t+\tau)].$$

Note that $r_{X,Y}(\tau) \neq r_{X,Y}(-\tau)$ but $r_{X,Y}(\tau) = r_{Y,X}(-\tau)$.
The complex-valued cross spectrum $R_{X,Y}(f)$ is defined so that

$$r_{X,Y}(\tau) = \int R_{X,Y}(f) e^{i2\pi f\tau} df.$$

The quadratic coherence spectrum is defined

$$\kappa^2_{X,Y}(f) = \frac{|R_{X,Y}(f)|^2}{R_X(f)R_Y(f)}, \;\; 0 \leq \kappa^2_{X,Y} \leq 1.$$

The same formulations apply for discrete time processes.

Differentiation

A weakly stationary process $X(t)$, $t \in \mathbb{R}$, is said to be differentiable in quadratic mean with the derivative $X'(t)$ if $r_X(\tau)$ is twice differentiable or if

$$V[X'(t)] = \int_{-\infty}^{\infty} (2\pi f)^2 R_X(f) df < \infty.$$

We have $m_{X'} = 0$,

$$r_{X'}(\tau) = -r''_X(\tau),$$

and

$$R_{X'}(f) = (2\pi f)^2 R_X(f).$$

The cross-covariance function,

$$r_{X,X'}(t, t+\tau) = r'_X(\tau),$$

which implies $r_{X,X'}(t+\tau, t) = -r'_X(\tau)$ and $r_{X,X'}(t,t) = 0$.

Exam 171028-3

A zero-mean, weakly stationary, Gaussian process $X(t)$, $t \in \mathbb{R}$ has the spectral density

$$R_X(f) = \pi e^{-2\pi|f|}.$$

State, with a short motivation, for each of the following statements if it is right or wrong. Each correct answer with motivation gives 2 credits.

a) The covariance function is $r_X(\tau) = \frac{1}{1+\tau^2}$.

b) The process is differentiable in quadratic mean.

c) The variance $V[(X(0) + X(t))/2]$ approaches zero 0 when $t \to \infty$.

d) After filtering through a linear filter with frequency function $H(f) = 1 + if$, the spectral density of the process is constant for all frequencies.

e) A new process $Y(t) = 3X(t) - 2X'(t)$ is created. The process $Y(t)$ is a Gaussian process.

(10p)

Optimal filters

- Mean square error optimal filter, minimize $E[(Y(t) - S(t))^2]$.
- The matched filter for binary detection is

$$h_{opt}(u) = s(T - u),$$

for the zero-mean white noise disturbance case. For equal decision errors, the decision level $k = s_{out}(T)/2$ with errors $\alpha = \beta = 1 - \Phi(\frac{s_{out}(T)}{2\sigma_N})$.

- The Wiener filter frequency function is

$$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)},$$

where $R_S(f)$ and $R_N(f)$ are the zero-mean signal and noise spectral densities respectively.

Exam 161222-6

In a simple model for stock prices it is assumed that the price of a certain stock, X_t, $t = 0, \pm 1, \pm 2, \ldots$, varies as a Gaussian stationary process, defined by the MA-process

$$X_t = e_t + e_{t-1} + e_{t-2},$$

where the uncorrelated random variables e_t have $E[e_t] = 0$ and $V[e_t] = 0.25$. Utilizing the available information up to time t, a prediction of X_{t+1} should be made as

$$\hat{X}_{t+1} = aX_t + bX_{t-1}.$$

Determine a and b so that

$$P(|X_{t+1} - \hat{X}_{t+1}| > 1),$$

is minimized.

(20p)

Sampling

The continuous time process $Y(t)$, $t \in \mathbb{R}$ is sampled to the discrete time sequence $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$. The covariance function is

$$r_Z(\tau) = r_Y(\tau), \;\; \tau = 0, \pm d, \pm 2d, \ldots$$

and the spectral density is found as

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) \;\; -f_s/2 < f \leq f_s/2.$$

with $f_s = 1/d$ as the sampling frequency.

Estimation of mean

If $X(t)$, $t = 1, 2\ldots$ is weakly stationary with the unknown expected value m then

$$\hat{m}_n = \frac{1}{n}\sum_{t=1}^{n} X(t)$$

is an unbiased estimate of m as $E[\hat{m}_n] = m$. The variance is

$$V[\hat{m}_n] = C[\frac{1}{n}\sum_{t=1}^{n} x(t), \frac{1}{n}\sum_{s=1}^{n} x(s)] = \frac{1}{n^2}\sum_{\tau=-n+1}^{n-1} (n - |\tau|) r(\tau).$$

For large n,

$$V[\hat{m}_n] \approx \frac{1}{n}\sum_{\tau} r(\tau).$$

Exam 161222-3

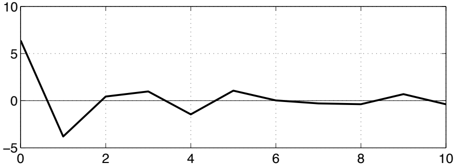
The average level m of a stationary stochastic process, $Y_t = m + X_t$, $t = 0, \pm1, \pm2, \ldots$, should be estimated. It is assumed that the process X_t is an MA(2)-process, and we know that it is either

$$X_t = e_t + 2e_{t-1} + e_{t-2}, \quad \text{Alt. 1}$$

or

$$X_t = e_t - 2e_{t-1} + e_{t-2}, \quad \text{Alt. 2}$$

where e_t, $t = 0, \pm1, \pm2 \ldots$, is white Gaussian noise with expected value zero and variance one. From 100 realizations, a covariance function estimate, $\hat{r}_x(\tau)$ is computed, (see the figure below). Determine and motivate which of the two suggested disturbance models above it is most likely to be. Compute the **true covariance function** of your choice and use this in the following calculations (cont. on next page)



Exam 161222-3, cont.

One can choose between two estimates for m,

$$\hat{m}_1 = \frac{Y_t + Y_{t-1} + Y_{t-2}}{3}$$

or

$$\hat{m}_2 = \frac{Y_t + Y_{t-2}}{2}.$$

Determine the variance of the most optimal estimator, $\hat{m}_1$ or $\hat{m}_2$.

(10p)

Spectrum estimation

The periodogram is defined as

$$\widehat{R}_x(f) = \frac{1}{n}|\sum_t x(t)w(t)e^{-i2\pi ft}|^2,$$

where $w(t)$ is a data window. With the Hanning window, the spectrum estimate will have better leakage properties (lower sidelobes), although the resolution is somewhat degraded, (wider mainlobe), in comparison to the rectangular window.

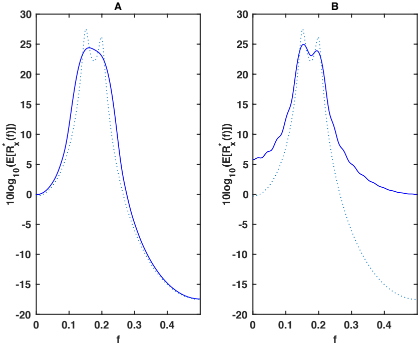
The variance of the periodogram is

$$V[\widehat{R}_x(f)] \approx R_X^2(f) \;\; 0 < |f| < 0.5$$

Dividing the sequence into K possibly overlapping sequences and calculating the average of K approximately uncorrelated estimates (Welch method), reduces the variance to

$$V[\widehat{R}_{mv}(f)] \approx \frac{1}{K} R_X^2(f).$$

Exam 180108-3



a) In the figures above, the expected values of the periodogram the rectangular and the Hanning window, calculated from *n* samples, are illustrated with solid lines. Which of the figures shows the expected value using the two windows. Motivate your answer.

b) The sequence is divided into *K* sequences and the final estimate is calculated as the average of *K* uncorrelated periodograms. How much does the variance decrease in comparison to the periodogram in a)?

(10p)