Stationary stochastic processes

Maria Sandsten

Lecture 10

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Last week: The MA(q)-process

A Moving Average, an MA(q)-process, is given by

$$X_t = c_0 e_t + c_1 e_{t-1} + \ldots + c_q e_{t-q},$$

where e_t, $0, \pm 1, \pm 2, \ldots$, is zero-mean white noise with variance σ^2. The expected value is $E[X_t] = 0$ and the covariance function is given from

$$r_X(\tau) = C[X_t, X_{t+\tau}],, \quad -q \leq \tau \leq q.$$

The spectral density is

$$R_X(f) = r_X(0) + 2\sum_{\tau=1}^{q} r_X(\tau)\cos(2\pi f \tau), \quad -1/2 < f \leq 1/2$$

or

$$R_X(f) = |1 + c_1 e^{-i2\pi f} + \ldots + c_q e^{-i2\pi f q}|^2 \sigma^2, \quad -1/2 < f \leq 1/2.$$

For an MA(q)-process we always have q complex conjugated zeros (if not on the real axis).

Last week: The AR(p)-process

An Auto-Regressive, an AR(p)-process is given by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t,$$

where e_t, $t = 0, \pm 1, \pm 2 \ldots$, is zero-mean white noise with variance σ^2. The expected value is $E[X_t] = 0$ and the covariance function is of infinite length and computed from the Yule-Walker equations defined as

$$r_X(\tau) + a_1 r_X(\tau - 1) + \ldots + a_p r_X(\tau - p) = \begin{cases} \sigma^2 & \text{for } \tau = 0 \\ 0 & \text{for } \tau = 1, 2, \ldots \end{cases}$$

The spectral density is

$$R_X(f) = \frac{\sigma^2}{|1 + a_1 e^{-i2\pi f} + \ldots + a_p e^{-i2\pi f p}|^2}, \quad -1/2 < f \leq 1/2.$$

For an AR(p)-process we always have p complex conjugated poles (if not on the real axis).

Schedule for todays lecture

- Mean squared error optimal filter. Example.
- Wiener filter: the optimal cleaning filter. Example.
- Matched filter: the optimal detection filter, including a statistical decision test. Example.

Example (old exam)

A discrete time process S_t, $t = 0, \pm 1, \pm 2, \ldots$ is disturbed by noise N_t, $t = 0, \pm 1, \pm 2, \ldots$. To improve the measured signal, $Z_t = S_t + N_t$ is filtered in a filter with frequency function

$$H(f) = a_0 + a_1\, e^{-i2\pi f},$$

giving the output process Y_t. Determine the coefficients a_0 and a_1 so that the mean squared error,

$$E\left[(Y_t - S_t)^2\right],$$

becomes as small as possible. The processes S_t and N_t are independent and have expected values, $m_S = m_N = 0$, and covariance functions $r_S(\tau) = 0.5^{|\tau|}$ and $r_N(\tau) = (-0.5)^{|\tau|}$, $\tau = 0, \pm 1, \pm 2, \ldots$.

(20p)

Solution: Exercise (old exam)

The output from the filter is

$$Y_t = a_0 Z_t + a_1 Z_{t-1} = a_0 S_t + a_1 S_{t-1} + a_0 N_t + a_1 N_{t-1},$$

and the mean squared error is

$$E\left[(Y_t - S_t)^2\right] = V\left[(a_0 S_t + a_1 S_{t-1} - S_t + a_0 N_t + a_1 N_{t-1})\right] =$$

$$a_0^2 V[S_t] + a_1^2 V[S_{t-1}] + a_0^2 V[N_t] + a_1^2 V[N_{t-1}] + V[S_t] +$$

$$2a_0 a_1 C[S_t, S_{t-1}] + 2a_0 a_1 C[N_t, N_{t-1}] - 2a_0 V[S_t] - 2a_1 C[S_t, S_{t-1}] =$$

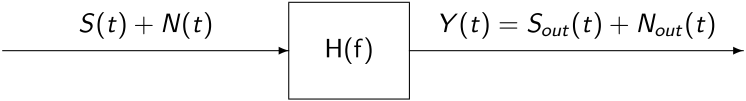
$$2a_0^2 + 2a_1^2 + 1 - 2a_0 - a_1 = \epsilon.$$

Differentiation gives

$$\frac{\partial \epsilon}{\partial a_0} = 4a_0 - 2 = 0 \quad \rightarrow a_0 = \frac{1}{2}, \quad \frac{\partial \epsilon}{\partial a_1} = 4a_1 - 1 = 0 \quad \rightarrow a_1 = \frac{1}{4},$$

and the second derivatives are both positive giving a minimum.

The Wiener filter



The Wiener filter is given by maximizing the Signal-to-Noise Ratio (SNR) defined as

$$\text{SNR} = \frac{\text{power for wanted signal}}{\text{power for not wanted signal}} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]},$$

where the zero-mean stationary process with spectral density $R_S(f)$ and the zero-mean stationary disturbance process with spectral density $R_N(f)$ are independent.

Proof: The Wiener filter

As $E[S(t)^2]$ is pre-defined, maximizing the SNR,

$$\text{SNR} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]},$$

is equal to minimizing the mean squared error,

$$\epsilon^2 = E[(Y(t) - S(t))^2] = V[(Y(t) - S(t))].$$

With $Y(t) = S_{out}(t) + N_{out}(t)$ we get,

$$\epsilon^2 = V[(S_{out}(t) + N_{out}(t) - S(t))] =$$

$$V[(S_{out}(t) - S(t))] + V[N_{out}(t)].$$

Proof: The Wiener filter

We find the following expression to be minimized,

$$\epsilon^2 = V[(S_{out}(t) - S(t))] + V[N_{out}(t)],$$
$$= \int (|H(f) - 1|^2 R_S(f) + |H(f)|^2 R_N(f)) df.$$

as $S_{out}(t)$ can be seen as the output from the filter with $S(t)$ as input giving

$$V[S_{out}(t) - S(t)] = V[\int h(u)S(t-u)du - \int \delta(u)S(t-u)du] =$$
$$= V[\int h_1(u)S(t-u)du] = \int |H_1(f)|^2 R_S(f) df,$$

where $h_1(u) = h(u) - \delta(u)$ and $H_1(f) = H(f) - 1$, and the variance of the filtered disturbance is,

$$V[N_{out}(t)] = \int |H(f)|^2 R_N(f) df.$$

Proof: The Wiener filter

We minimize independently for every f so omitting f above gives,

$$\min_H[|H-1|^2 R_S + |H|^2 R_N] = \min_H[(H-1)(\bar{H}-1)R_S + H\bar{H}R_N] =$$

$$= \min_H[H\bar{H}(R_S+R_N) - (H+\bar{H})R_S + R_S] =$$

$$= \min_H[(\Re H^2 + \Im H^2)(R_S+R_N) - 2\Re H R_S + R_S].$$

If $\Im H = 0$ the imaginary part is minimized to zero and therefore H is real-valued for all f. Minimize the real-valued expression

$$\min_H[H^2(R_S+R_N) - 2HR_S + R_S],$$

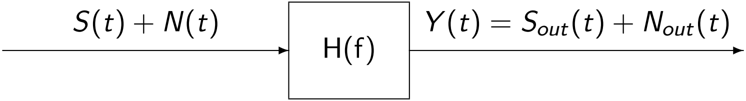
where the derivative with respect to H is

$$2H(R_s+R_N) - 2R_S = 0,$$

i.e.,

$$H_{opt} = \frac{R_S}{R_S+R_N}.$$

Summary: The Wiener filter



The Wiener filter, which maximizes the Signal-to-Noise Ratio,

$$\mathrm{SNR}_{max} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]},$$

has the real-valued frequency function

$$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)}.$$

A Wiener filter example

A zero-mean stationary stochastic process $S(t)$, $t \in \mathbb{R}$, is disturbed by a zero-mean noise process, $N(t)$, $t \in \mathbb{R}$. The two processes are independent. The covariance function for the signal is

$$r_S(\tau) = e^{-|\tau|}$$

and for the noise

$$r_N(\tau) = \alpha e^{-\beta|\tau|}.$$

Calculate the frequency function of the Wiener filter that maximizes the SNR

$$\text{SNR} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]}.$$

(10p)

Solution: A Wiener filter example

The Wiener filter is given by

$$H(f) = \frac{R_S(f)}{R_S(f) + R_N(f)}$$

where

$$R_S(f) = \frac{2}{1 + (2\pi f)^2},$$

and

$$R_N(f) = \alpha \frac{2\beta}{\beta^2 + (2\pi f)^2}.$$

We get

$$H(f) = \frac{\beta^2 + 4\pi^2 f^2}{\beta^2 + \alpha\beta + 4\pi^2(1 + \alpha\beta) f^2}.$$

The matched detector

In digital communication, a signal $s(t)$, $0 \leq t \leq T$, is either sent, i.e. is 'one', or not sent, i.e. is 'zero'. At the receiver, a filter is applied so the signal is to be correctly verified as a 'one' or a 'zero'. We apply a causal filter, $h(t)$, $0 \leq t \leq T$, which will give as output at $t = T$,

$$s_{out}(T) + N_{out}(T) = \int_0^T h(u)s(T-u)du + \int_0^T h(u)N(T-u)du,$$

if signal is sent, and just

$$N_{out}(T) = \int_0^T h(u)N(T-u)du,$$

if signal is not sent. The disturbance is usually assumed to be stationary Gaussian noise.

Proof and summary: The matched filter for white noise disturbance

If signal is sent, the SNR at $t = T$ should be maximized, i.e.

$$\max \; SNR = \frac{s_{out}^2(T)}{V[N_{out}(T)]}$$

For the case of white noise disturbance, with input variance σ_N^2, the SNR is

$$SNR = \frac{(\int_0^T h(u)s(T-u)du)^2}{\sigma_N^2 \int_0^T h^2(u)du}.$$

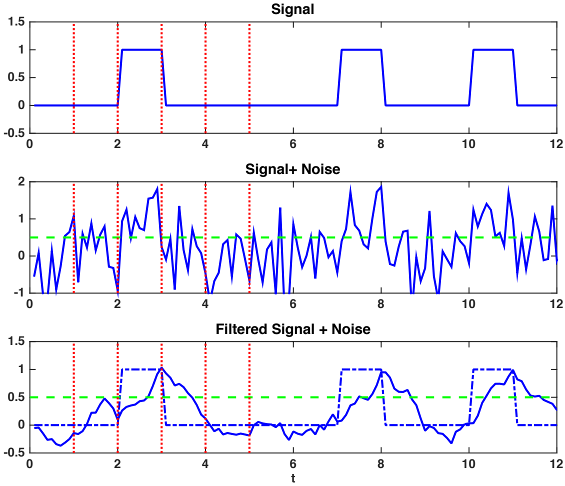
The maximum is reached for the matched filter

$$h_{opt}(t) = s(T-t), \quad 0 \le t \le T,$$

with the maximized SNR as

$$SNR_{max} = \frac{\int_0^T s^2(T-u)du}{\sigma_N^2}.$$

Matched filter example



Summary matched filter: coloured noise

Theorem 8.6: The causal matched filter for detection of $s(t)$, $0 \leq t \leq T$, disturbed by zero-mean coloured noise with covariance function $r_N(\tau)$, has the impulse response $h(t)$, $0 \leq t \leq T$, found from the solution of

$$s(T-t) = \int_0^T h(u) r_N(t-u) du.$$

Proof, see pages 217-218.

Statistical decision test

We formulate a statistical decision problem with the matched filter as a hypothesis test with Gaussian distributions:

$$
\begin{aligned}
H_0: \quad Y(T) &= N_{out}(T), \quad \in \mathrm{N}(0, \sigma^2_{Nout}), \\
H_1: \quad Y(T) &= s_{out}(T) + N_{out}(T), \quad \in \mathrm{N}(s_{out}(T), \sigma^2_{Nout}),
\end{aligned}
$$

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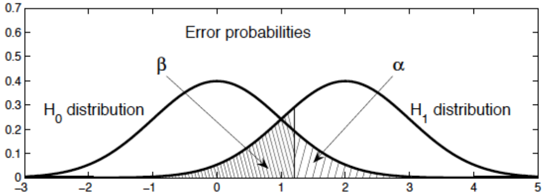

Figure 8.7: *Filter output distributions under two hypotheses.*

Equal probability for error decisions

The probability for error decisions are,

$$\begin{aligned} \alpha &= P(Y(T) > k | Y(T) \in \mathrm{N}(0, \sigma^2_{Nout})) = 1 - \Phi(\frac{k}{\sigma_{Nout}}) = \Phi(\frac{-k}{\sigma_{Nout}}), \\ \beta &= P(Y(T) < k | Y(T) \in \mathrm{N}(s_{out}(T), \sigma^2_{Nout})) = \Phi(\frac{k - s_{out}(T)}{\sigma_{Nout}}). \end{aligned}$$

If the errors should be equal, $\alpha = \beta$, then $k - s_{out}(T) = -k$, i.e. the decision level is

$$k = s_{out}(T)/2,$$

yielding the probability of errors,

$$\alpha = \beta = 1 - \Phi(\frac{s_{out}(T)}{2\sigma_{Nout}}).$$

Example

We would like to detect the sent signal, $s(t) = 1$, $t = 0, 1, 2$, where the zero-mean Gaussian disturbance is $N(t)$, $t = 0, \pm 1, \pm 2 \ldots$.

a) Determine the decision level and the probability of errors, using the causal matched filter for white noise disturbance with variance $\sigma_N^2 = 1$.

b) Calculate the causal matched filter, the decision level and probability of errors, when the noise covariance function is $r_N(0) = 1$, $r_N(\pm 1) = 0.5$.

The decision level should, in both cases, be decided so the two error probabilities are equal.

Solution a)

The matched filter for white noise disturbance is $h(t) = s(T - t)$ where the maximum SNR is found at $T = 2$ as

$$s_{out}(2) = \sum_{u=0}^{2} h(u)s(2-u) = \sum_{u=0}^{2} s^2(2-u) = 3.$$

Choose the decision level $k = \frac{s_{out}(2)}{2} = 1.5$ for equal errors. We get

$$\sigma^2_{Nout} = \sigma^2_N \sum_{u=0}^{2} h^2(u) = 3.$$

Then

$$P(Y(2) > 1.5 | Y(2) \in \mathrm{N}(0,3)) = 1 - \Phi(\frac{1.5}{\sqrt{3}}) = 0.196,$$

$$P(Y(2) < 1.5 | Y(2) \in \mathrm{N}(3,3)) = \Phi(\frac{1.5-3}{\sqrt{3}}) = 0.196.$$

Solution b)

The optimal causal filter fulfills the equation system for $T = 2$,

$$s(2-t) = \sum_{u=0}^{2} h(u) r_N(t-u),$$

which has the solution $h(0) = h(2) = 1, h(1) = 0$. We get $s_{out}(2) = \sum_{u=0}^{2} h(u)s(2-u) = 2$. Choose decision level $k = \frac{s_{out}(2)}{2} = 1$ for equal errors. We get

$$\sigma_{Nout}^2 = \sum_{u=0}^{2} \sum_{v=0}^{2} h(u)h(v) r_N(u-v) = 2,$$

and

$$P(Y(2) > 1 | Y(2) \in \mathrm{N}(0,2)) = 1 - \Phi(\frac{1}{\sqrt{2}}) = 0.239,$$

$$P(Y(2) < 1 | Y(2) \in \mathrm{N}(2,2)) = \Phi(\frac{1-2}{\sqrt{2}}) = 0.239.$$