Stationary stochastic processes

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Lecture 10

October 2 2018

The MA(q)-process

A Moving Average, an MA(q)-process, is given by

$$X_t = c_0 e_t + c_1 e_{t-1} + \ldots + c_q e_{t-q},$$

where e_t, $0, \pm1, \pm2, \ldots,$ is zero-mean white noise with variance σ^2. The expected value is $E[X_t] = 0$ and the covariance function is given from

$$r_X(\tau) = C[X_t, X_{t+\tau}].$$

The spectral density is

$$R_X(f) = r_X(0) + 2\sum_{\tau=1}^{q} r_X(\tau)\cos(2\pi f\tau) = |1 + c_1 e^{-i2\pi f} + \ldots + c_q e^{-i2\pi fq}|^2\sigma^2,$$

for $-1/2 < f \leq 1/2$. For an MA(q)-process we always have q zeros (possibly complex conjugated).

The AR(p)-process

An Auto-Regressive, an AR(p)-process is given by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t,$$

where, e_t, $t = 0, \pm 1, \pm 2 \ldots$, is zero-mean white noise with variance σ^2. The expected value is $E[X_t] = 0$ and the covariance function is computed from the Yule-Walker equations defined as

$$r_X(\tau) + a_1 r_X(\tau - 1) + \ldots + a_p r_X(\tau - p) = \begin{cases} \sigma^2 & \text{for } \tau = 0 \\ 0 & \text{for } \tau = 1, 2, \ldots \end{cases}$$

The spectral density is

$$R_X(f) = \frac{\sigma^2}{|1 + a_1 e^{-i2\pi f} + \ldots + a_p e^{-i2\pi f p}|^2}, \quad -1/2 < f \leq 1/2.$$

For an AR(p)-process we always have p poles (possibly complex conjugated).

Computer exercise 3

More AR- and MA-processes in computer exercise 3!

News HT 2018

The registration system opens for Computer exercise 3, October 8 8.00.

Answer the questions in 1.1 of the tutorial of Comp. exercise 3 and E-mail your answers to FMSF10@matstat.lu.se at latest by **Thursday morning October 11 at 8.00**.

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Comp. exercise 2:

- Tue Oct 2 13.15-15.00 (E:Hacke, E:Panter, E:Val)
- Wed Oct 3 10.15-12.00 (E:Satu, E:Uran)
- Fri Oct 5 15.15-17.00 (E:Elg, E:Elgkalv)

Comp. exercise 3:

- Mon Oct 15 13.15-15.00 (E:Satu, E:Uran)
- Tue Oct 16 13.15-15.00 (E:Hacke, E:Panter, E:Val)
- Fri Oct 19 15.15-17.00 (E:Elg, E:Elgkalv)

Schedule for todays lecture

- Mean squared error optimal filter: an example.
- Wiener filter: the optimal cleaning filter.
- Matched filter: the optimal detection filter.

Example (old exam)

A discrete time process $S_t,\ t = 0, \pm 1, \pm 2, \dots$ is disturbed by noise $N_t,\ t = 0, \pm 1, \pm 2, \dots$. To improve the measured signal, $Z_t = S_t + N_t$ is filtered in a filter with frequency function

$$H(f) = a_0 + a_1\ e^{-i2\pi f},$$

giving the output process Y_t. Determine the coefficients a_0 and a_1 so that the mean squared error, $E\left[(Y_t - S_t)^2\right]$, becomes as small as possible. The processes S_t and N_t are independent and have expected values, $m_S = m_N = 0$, and covariance functions $r_S(\tau) = 0.5^{|\tau|}$ and $r_N(\tau) = (-0.5)^{|\tau|}$, $\tau = 0, \pm 1, \pm 2, \dots$.

(20p)

Exercise (old exam)

The output from the filter is

$$Y_t = a_0 Z_t + a_1 Z_{t-1},$$

and the mean squared error is

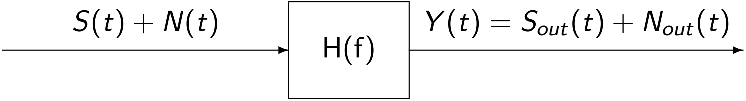
$$E\left[(a_0 Z_t + a_1 Z_{t-1} - S_t)^2\right] =$$
$$E\left[(a_0 S_t + a_1 S_{t-1} - S_t + a_0 N_t + a_1 N_{t-1})^2\right] =$$
$$a_0^2 V[S_t] + a_1^2 V[S_{t-1}] + a_0^2 V[N_t] + a_1^2 V[N_{t-1}] + V[S_t] +$$
$$2a_0 a_1 C[S_t, S_{t-1}] + 2a_0 a_1 C[N_t, N_{t-1}] - 2a_0 V[S_t] - 2a_1 C[S_t, S_{t-1}] =$$
$$2a_0^2 + 2a_1^2 + 1 - 2a_0 - a_1 = \epsilon.$$

Differentiation gives

$$\frac{\partial \epsilon}{\partial a_0} = 4a_0 - 2 = 0 \quad \to a_0 = \frac{1}{2}$$
$$\frac{\partial \epsilon}{\partial a_1} = 4a_1 - 1 = 0 \quad \to a_1 = \frac{1}{4}$$

The Wiener filter

A zero-mean stationary process with spectral density $R_S(f)$ and a zero-mean stationary disturbance process with spectral density $R_N(f)$ are independent.



The Wiener filter maximizes the Signal-to-Noise Ratio (SNR) defined as

$$\text{SNR} = \frac{\text{power for wanted signal}}{\text{power for not wanted signal}} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]}.$$

The Wiener filter

The Wiener filter has the real-valued frequency function

$$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)},$$

which is given by maximized SNR,

$$\mathrm{SNR}_{opt} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]},$$

given by minimizing the mean squared error,

$$\epsilon^2 = E[(Y(t) - S(t))^2].$$

The Wiener filter

We get,

$$\epsilon^2 = E[(Y(t) - S(t))^2] = E[(S_{out}(t) + N_{out}(t) - S(t))^2] =$$

$$E[(S_{out}(t) - S(t))^2] + E[(S_{out}(t) - S(t))N_{out}(t)] + E[N_{out}(t)^2] =$$

$$E[(S_{out}(t) - S(t))^2] + E[N_{out}(t)^2] =$$

$$V[(S_{out}(t) - S(t))] + V[N_{out}(t)].$$

The variance of the filtered disturbance is,

$$V[N_{out}(t)] = \int |H(f)|^2 R_N(f) df.$$

Wiener filter

$S_{out}(t)$ can be seen as the output from the filter with $S(t)$ as input giving

$$V[S_{out}(t) - S(t)] = V[\int h(u)S(t-u)du - \int \delta(u)S(t-u)du] =$$

$$= V[\int h_1(u)S(t-u)du] = \int |H_1(f)|^2 R_S(f)df,$$

where $h_1(u) = h(u) - \delta(u)$ and $H_1(f) = H(f) - 1$. We find the following expression to be minimized,

$$\begin{aligned} \epsilon^2 &= E[(Y(t) - S(t))^2] = \\ &= E[(S_{out}(t) - S(t))^2] + E[N_{out}(t)^2] = \\ &= \int (|H(f) - 1|^2 R_S(f) + |H(f)|^2 R_N(f))df. \end{aligned}$$

The Wiener filter

Omitting f,

$$\min_H[(H-1)(\bar{H}-1)R_S + H\bar{H}R_N] =$$
$$= \min_H[H\bar{H}(R_S+R_N) - (H+\bar{H})R_S + R_S] =$$
$$= \min_H[(\Re H^2 + \Im H^2)(R_S+R_N) - 2\Re H R_S + R_S].$$

If $\Im H = 0$ the imaginary part of the expression is zero and we know that H is real-valued for all f.
Minimize

$$\min_H[H^2(R_S+R_N) - 2HR_S + R_S],$$

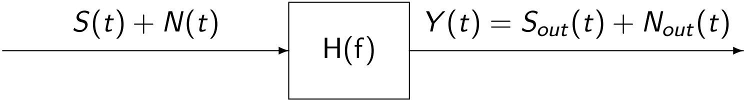
where the derivative with respect to H is

$$2H(R_s+R_N) - 2R_S = 0,$$

i.e.,

$$H_{opt} = \frac{R_S}{R_S+R_N}.$$

The Wiener filter



The Wiener filter, which maximizes the Signal-to-Noise Ratio,

$$\text{SNR}_{opt} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]},$$

has the real-valued frequency function

$$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)}.$$

Exercise (old exam) modified

A stationary process $S(t), t \in \mathbb{R}$ with spectral density

$$R_S(f) = |f|, \ \ |f| < 2,$$

is disturbed by a stationary process, $N(t), t \in \mathbb{R}$ with spectral density

$$R_N(f) = 1 - |f|, \ \ |f| < 1.$$

The processes $S(t)$ and $N(t)$ are both zero-mean and independent.

a) Calculate the SNR $= E[S^2(t)]/E[(Y(t) - S(t))^2]$, where $Y(t)$ is the output from the filter $H(f) = 1$, $1 \leq |f| \leq 2$, and the input is $S(t) + N(t)$.

b) Calculate the Wiener filter and the corresponding SNR.

Solution a)

The SNR is defined as

$$\text{SNR} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]},$$

where

$$E[S^2(t)] = V[S(t)] = 2\int_0^2 R_S(f)df = 4,$$

and

$$E[(Y(t) - S(t))^2] = E[(S_{out}(t) - S(t))^2] + E[N_{out}^2(t)] =$$

$$2\int_0^2 |H(f) - 1|^2 R_S(f)df + 0 = 2\int_0^1 R_S(f)df = 1.$$

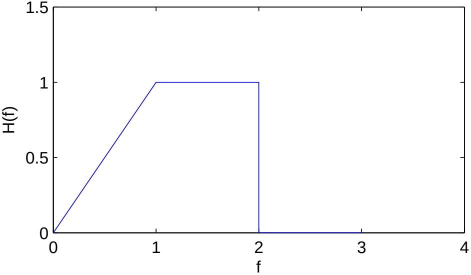
The resulting SNR $= 4$.

Solution b)

The Wiener filter, given from

$$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)},$$

is defined as



Solution b)

The power of the wanted signal,

$$E[S^2(t)] = V[S(t)] = 2\int_0^2 R_S(f)df = 4,$$

the same as in a). The power of the not wanted signal, is the sum of

$$E[(S_{out}(t) - S(t))^2] = 2\int_0^2 |H(f)-1|^2 R_S(f)df = 2\int_0^1 (1-f)^2 f\ df = \frac{1}{6},$$

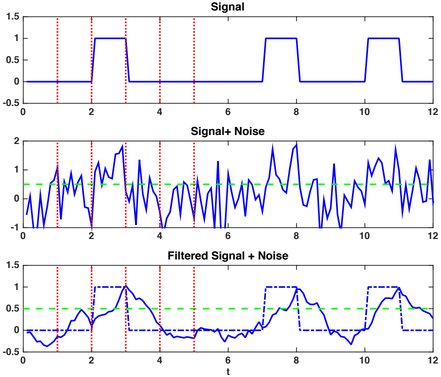
and

$$E[N_{out}^2(t)] = 2\int_0^1 |H(f)|^2 R_N(f)df = 2\int_0^1 f^2(1-f)df = \frac{1}{6}.$$

The resulting $\text{SNR}_{opt} = 4/(\frac{1}{6} + \frac{1}{6}) = 12$.

The matched detector

In digital communication, a signal $s(t) = 1$, $0 \leq t \leq T$, is either sent, i.e. is 'one', or not sent, i.e. is 'zero'. At the receiver, a matched detector should filter the signal to be correctly verified as a 'one' or a 'zero'.



The matched detector

At the receiver we apply a causal filter, $h(t)$, $0 \leq t \leq T$, which will give as output at $t = T$,

$$s_{out}(T) + N_{out}(T) = \int_0^T h(u)s(T-u)du + \int_0^T h(u)N(T-u)du,$$

if signal is sent, and just

$$N_{out}(T) = \int_0^T h(u)N(T-u)du,$$

if no signal is sent. The disturbance is usually assumed to be stationary.

The matched detector

The SNR at $t = T$ should be maximized, i.e.

$$\max \ SNR = \frac{s_{out}^2(T)}{V[N_{out}(T)]}$$

For the case of white noise disturbance, with input variance σ_N^2, the SNR is

$$SNR = \frac{(\int_0^T h(u)s(T-u)du)^2}{\sigma_N^2 \int_0^T h^2(u)du}.$$

The maximum is reached for the matched filter

$$h_{opt}(t) = s(T-t), \quad 0 \leq t \leq T,$$

and the optimal SNR is

$$SNR_{max} = \frac{\int_0^T s^2(T-u)du}{\sigma_N^2}.$$

Statistical decision test

We formulate a statistical decision problem with the matched filter as a hypothesis test with Gaussian distributions:

$$
\begin{aligned}
H_0: \quad Y(T) &= N_{out}(T), \quad \in \mathrm{N}(0, \sigma^2_{Nout}), \\
H_1: \quad Y(T) &= s_{out}(T) + N_{out}(T), \quad \in \mathrm{N}(s_{out}(T), \sigma^2_{Nout}),
\end{aligned}
$$

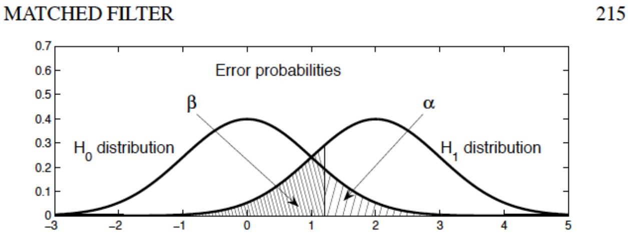


Figure 8.7: *Filter output distributions under two hypotheses.*

Equal probability for error decisions

The probability for error decisions are,

$$\alpha = P(Y(T) > k | Y(T) \in \mathrm{N}(0, \sigma_{Nout}^2)) = 1 - \Phi(\frac{k}{\sigma_{Nout}}) = \Phi(\frac{-k}{\sigma_{Nout}}),$$
$$\beta = P(Y(T) < k | Y(T) \in \mathrm{N}(s_{out}(T), \sigma_{Nout}^2)) = \Phi(\frac{k - s_{out}(T)}{\sigma_{Nout}}).$$

If the errors should be equal, $\alpha = \beta$, then $k - s_{out}(T) = -k$, i.e. the decision level is

$$k = s_{out}(T)/2,$$

yielding the probability of errors,

$$\alpha = \beta = 1 - \Phi(\frac{s_{out}(T)}{2\sigma_{Nout}}).$$

Example

We would like to detect the sent signal, $s_t = 1$, $t = 0, 1, 2$, where the zero-mean Gaussian disturbance is N_t.

a) Determine the decision level and the probability of errors, using the causal matched filter for white noise disturbance with variance $\sigma_N^2 = 1$.

b) Calculate the causal matched filter, the decision level and probability of errors, when the noise covariance function is $r_N(0) = 1$, $r_N(\pm 1) = 0.5$.

The decision level should, in both cases, be decided so the two error probabilities are equal.

Solution a)

We get for $t = T = 2$:

$$sout_2 = \sum_{u=0}^{2} h(u) s_{2-u} = \sum_{u=0}^{2} s_{2-u}^2 = 3.$$

Choose decision level $k = \frac{sout_2}{2} = 1.5$ for equal errors. We get

$$\sigma_{Nout}^2 = \sigma_N^2 \sum_{u=0}^{2} h^2(u) = 3.$$

Then

$$P(Y(2) > 1.5 | Y(2) \in \mathrm{N}(0,3)) = 1 - \Phi(\frac{1.5}{\sqrt{3}}) = 0.196,$$

$$P(Y(2) < 1.5 | Y(2) \in \mathrm{N}(3,3)) = \Phi(\frac{1.5 - 3}{\sqrt{3}}) = 0.196.$$

Summary matched filter: coloured noise

Theorem 8.6: The causal matched filter for detection of $s(t)$, $0 \leq t \leq T$, disturbed by zero-mean coloured noise with covariance function $r_N(\tau)$, has the impulse response $h(t)$, $0 \leq t \leq T$, found from the solution of

$$s(T-t) = \int_0^T h(u) r_N(t-u) du.$$

Proof, see pages 217-218.

Solution b)

The optimal causal filter fulfills the equation system for $T = 2$,

$$s_{2-t} = \sum_{u=0}^{2} h(u) r_N(t-u),$$

which has the solution $h(0) = h(2) = 1, h(1) = 0$. We get $sout_2 = \sum_{u=0}^{2} h(u) s_{2-u} = 2$. Choose decision level $k = \frac{sout_2}{2} = 1$ for equal errors. We get

$$\sigma^2_{Nout} = \sum_{u=0}^{2} \sum_{v=0}^{2} h(u) h(v) r_N(u-v) = 2,$$

and

$$P(Y(2) > 1 | Y(2) \in \mathrm{N}(0,2)) = 1 - \Phi(\frac{1}{\sqrt{2}}) = 0.239,$$

$$P(Y(2) < 1 | Y(2) \in \mathrm{N}(2,2)) = \Phi(\frac{1-2}{\sqrt{2}}) = 0.239.$$