Exercise

A stationary process is described by

$$4X_t - 4X_{t-1} + 2U_{t-1} + 2Z_{t-4} = 2 \cdot (V_{t-4} - X_{t-2}), \tag{1}$$

where U_t is white Gaussian noise and V_t is an MA(1)-process defined with $b_0 = 1$ and $b_1 = 3$ and white Gaussian noise as the input signal Z_t. The processes U_t and Z_t are independent and have variances $\sigma^2 = 1$ and expected values zero.

a) Show that the process can be written in the form of an AR(2)-process, i.e.,

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} = e_t, \tag{2}$$

where e_t is white Gaussian noise. Determine a_1, a_2 and the variance, $V[e_t]$.

b) Calculate the covariance function $r_X(\tau)$, for $\tau = 0, \pm 1, \pm 2, \pm 3$.

Solution

a) The MA(1)-process is defined as

$$V_t = Z_t + 3Z_{t-1},$$

and is inserted into Eq. (1), and when all variables of X_t are ordered on the left side and all other variables on the right side, we get

$$4X_t - 4X_{t-1} + 2X_{t-2} = 2Z_{t-4} + 6Z_{t-5} - 2Z_{t-4} - 2U_{t-1},$$

which fortunately can be reduced to

$$X_t - X_{t-1} + \frac{1}{2}X_{t-2} = \frac{3}{2}Z_{t-5} - \frac{1}{2}U_{t-1}. \tag{3}$$

If the variable Z_{t-4} had not been cancelled, the resulting process would have been an ARMA-process. Comparison of Eq. (3) with Eq. (2) gives $a_1 = -1$ and $a_2 = \frac{1}{2}$. The processes Z_t and U_t are independent white noise processes and therefore the right side of Eq. (3) will give a new resulting white noise process with variance

$$V[e_t] = (\frac{3}{2})^2 V[Z_{t-5}] + (\frac{1}{2})^2 V[U_{t-1}] = \frac{9}{4} + \frac{1}{4} = \frac{5}{2}.$$

For solution of b) see next page.

b) From a) the AR(2)-process is defined as

$$X_t - X_{t-1} + \frac{1}{2}X_{t-2} = e_t,$$

where e_t is white Gaussian noise with $V[e_t] = \frac{5}{2}$. The Yule-Walker-equations for $k = 0, 1, 2$ are given as,

$$\begin{aligned} r_X(0) - r_X(1) + \frac{1}{2}r_X(2) &= \frac{5}{2}, \quad (A1) \\ r_X(1) - r_X(0) + \frac{1}{2}r_X(1) &= 0, \quad (B1) \\ r_X(2) - r_X(1) + \frac{1}{2}r_X(0) &= 0. \quad (C1) \end{aligned}$$

Equation (B1) is simplified to

$$\frac{3}{2}r_X(1) - r_X(0) = 0, \quad (B2)$$

which gives $r_X(0) = \frac{3}{2}r_X(1)$ to be inserted into (A1) and (C1) yielding

$$\begin{aligned} \frac{3}{2}r_X(1) - r_X(1) + \frac{1}{2}r_X(2) &= \frac{5}{2}, \quad (A2) \\ r_X(2) - r_X(1) + \frac{1}{2} \cdot \frac{3}{2}r_X(1) &= 0. \quad (C2) \end{aligned}$$

(A2) and (C2) are simplified into

$$\begin{aligned} \frac{1}{2}r_X(1) + \frac{1}{2}r_X(2) &= \frac{5}{2}, \quad (A3) \\ r_X(2) - \frac{1}{4}r_X(1) &= 0. \quad (C3) \end{aligned}$$

With $r_X(2) = \frac{1}{4}r_X(1)$ given from (C3) inserted into (A3), we get

$$\frac{1}{2}r_X(1) + \frac{1}{2} \cdot \frac{1}{4}r_X(1) = \frac{5}{8}r_X(1) = \frac{5}{2}, \quad (A4)$$

resulting in $r_X(1) = 4$. Inserting $r_X(1) = 4$ into (C3) gives $r_X(2) = 1$ and into (B2) gives $r_X(0) = 6$. To find $r_X(3)$ we use the Yule-Walker equation $r_X(k) - r_X(k-1) + \frac{1}{2}r_X(k-2) = 0$ with $k = 3$ which yields,

$$r_X(3) = r_X(2) - \frac{1}{2}r_X(1) = -1.$$

Symmetry gives

$$\begin{aligned} r_X(0) &= 6 \\ r_X(\pm 1) &= 4 \\ r_X(\pm 2) &= 1 \\ r_X(\pm 3) &= -1. \end{aligned}$$