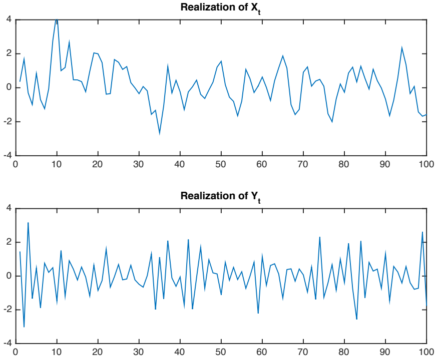
Example smoothing and filtering

The following exercise is a simplified illustration of the similarity between smoothing and filtering. Let X_t and Y_t be zero-mean discrete time stationary processes. The covariance functions are $r_X(0) = 1$, $r_X(\pm 1) = 1/2$ and $r_Y(0) = 1$, $r_Y(\pm 1) = -1/2$. Examples of realizations of the two different processes are seen below.



Example smoothing and filtering

A causal moving average filter with coefficients $h(0) = 1/2$ and $h(1) = 1/2$ is used to filter the two processes giving two new processes

$$Z_t = h(0)X_t + h(1)X_{t-1}, \quad W_t = h(0)Y_t + h(1)Y_{t-1}.$$

The filter $h(t)$ is a lowpass filter, a filter that passes low frequencies and reduces high frequencies. Can you without any calculations say which of the two processes that will have the most reduced variance after the filtering?

Compute the frequency function $H(f)$ and the amplitude function $|H(f)|$ to verify that the filter is a lowpass filter and calculate the variances, $r_Z(0)$ and $r_W(0)$.

Solution

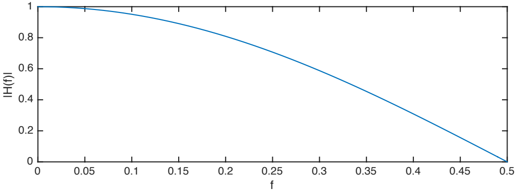
The frequency function is

$$H(f) = \sum_{t=-\infty}^{\infty} h(t)e^{-i2\pi ft} = h(0)e^{-i2\pi f0} + h(1)e^{-i2\pi f1} = \frac{1}{2} + \frac{1}{2}e^{-i2\pi f}.$$

The amplitude function is

$$|H(f)| = \sqrt{H(f)H^*(f)} = \sqrt{(\frac{1}{2} + \frac{1}{2}e^{-i2\pi f})(\frac{1}{2} + \frac{1}{2}e^{i2\pi f})} = \sqrt{\frac{1}{2} + \frac{1}{2}\cos(2\pi f)},$$

which is depicted in the figure below. Low frequencies have an amplification close to one (passed) where high frequencies have lower amplification (reduced).



Solution

We can compute the variance from

$$r_Z(0) = \int_{-1/2}^{1/2} |H(f)|^2 R_X(f) df.$$

In this case where there are few coefficients in both the impulse response and the covariance functions the variance can also be calculated from

$$r_Z(\tau) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h(u)h(v)r_X(\tau + u - v),$$

with $\tau = 0$ as

$$r_Z(0) = h(0)h(0)r_X(0) + h(1)h(0)r_X(1) + h(0)h(1)r_X(-1) + h(1)h(1)r_X(0).$$

Solution

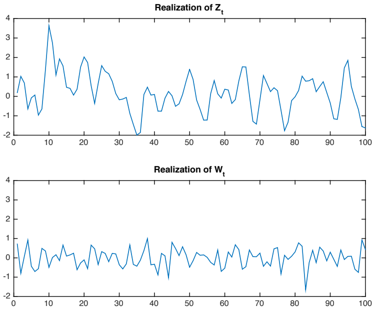
With $h(0) = h(1) = 1/2$ and $r_X(-1) = r_X(1)$ the expression is simplified into,

$$r_Z(0) = \frac{1}{2}(r_X(0) + r_X(1)) = \frac{1}{2}(1 + \frac{1}{2}) = \frac{3}{4}.$$

Similarly

$$r_W(0) = \frac{1}{2}(r_Y(0) + r_Y(1)) = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}.$$

Accordingly, the variance of W_t is smaller than for Z_t, which is also shown in the figure below of the output realizations.



Solution

We study the input spectral densities $R_X(f)$ and $R_Y(f)$ in the upper figure below. The process X_t has more low frequencies and Y_t consists of more high frequencies. The output spectral densities $R_Z(f) = |H(f)|^2 R_X(f)$ and $R_W(f) = |H(f)|^2 R_Y(f)$ are depicted in the lower figure where the low frequency content of X_t is kept in Z_t. The high frequency content of Y_t however is reduced by the lowpass filter and the total spectral density of W_t (and the variance) is accordingly very low.