A Gaussian stationary process $X(t)$, $t \in \mathbb{R}$, has covariance function $r_X(\tau) = \exp(-\tau^2/2)$. Calculate $P(X'(t) > 1)$ using the approximation

$$P\left( \frac{X(t + h/2) - X(t - h/2)}{h} > 1 \right),$$

where $h = 1, 0.5$ and $0.1$. 
Solution

Define a new Gaussian process

\[ Y(t) = \frac{X(t + \frac{h}{2}) - X(t - \frac{h}{2})}{h}. \]

The mean value is

\[ m_Y = E[Y(t)] = \frac{1}{h}(E[X(t + \frac{h}{2})] - E[X(t - \frac{h}{2})]) = \frac{1}{h}(m_X - m_X) = 0. \]

(Note that \( m_X \) can be unknown.) The variance is

\[ r_Y(0) = \frac{1}{h^2}(V[X(t + \frac{h}{2})] + V[X(t - \frac{h}{2})] - 2C[X(t + \frac{h}{2}), X(t - \frac{h}{2})]) = \]

\[ = \frac{1}{h^2}(r_X(0) + r_X(0) - 2r_X(h)) = \frac{2}{h^2}(1 - e^{-h^2/2}). \]
Solution

The process \( Y(t) \in N(0, \frac{2}{h^2}(1 - e^{-h^2/2})) \) and

\[
P(Y(t) > 1) = 1 - \Phi\left(\frac{1 - m_Y}{\sqrt{r_Y(0)}}\right) = 1 - \Phi\left(1/\sqrt{\frac{2}{h^2}(1 - e^{-h^2/2})}\right).
\]

- For \( h = 1 \), \( P(Y(t) > 1) = 1 - \Phi(1/\sqrt{2(1 - e^{-1/2})}) = 1 - \Phi(1.13) = 1 - 0.871 = 0.129 \)
- For \( h = 0.5 \), \( P(Y(t) > 1) = 1 - \Phi(1.03) = 1 - 0.848 = 0.152 \)
- For \( h = 0.1 \), \( P(Y(t) > 1) = 1 - \Phi(1.00) = 1 - 0.841 = 0.159 \)

The difference approximation approaches the derivative,
\( P(X'(t) > 1) = 0.159 \), (to be shown in lecture 9!).