Sampling exercise

A stationary continuous time process $Y(t),\ t \in \mathbb{R}$, has the spectral density

$$R_Y(f) = \begin{cases} 1 - |f| & |f| \leq 1 \\ 0 & |f| > 1. \end{cases}$$

a) Determine the covariance function of the process.

b) The process $Y(t)$ is sampled with $f_s = 3$, giving a the discrete time process $Z_t,\ t = 0 \pm d, \pm 2d, \ldots$. Determine the spectral density for the sampled process Z_t.

c) The process $Y(t)$ is now sampled with $f_s = 3/2$ and another discrete time process $Z_t,\ t = 0 \pm d, \pm 2d, \ldots$, is received. Determine the spectral density for the sampled process Z_t.

d) Finally, if instead $Y(t)$ is sampled with $f_s = 1$, determine the spectral density **and the covariance function** of the sampled process $Z_t,\ t = 0 \pm d, \pm 2d, \ldots$,

a) Solution

$\frac{\sin(2\pi\alpha\tau)}{2\pi\tau} \quad \text{if} \quad \tau \neq 0$	$0 \quad \text{if} \quad \lvert f \rvert > \alpha$
$\begin{cases} 1-\alpha\lvert\tau\rvert & \text{if} \quad \lvert\tau\rvert \leq \frac{1}{\alpha} \\ 0 & \text{if} \quad \lvert\tau\rvert > \frac{1}{\alpha} \end{cases}$	$\begin{cases} \frac{1}{\alpha} & \text{if} \quad f = 0 \\ \frac{2\alpha}{(2\pi f)^2}\left(1-\cos\left(\frac{2\pi f}{\alpha}\right)\right) & \text{if} \quad f \neq 0 \end{cases}$
$g(\tau)h(\tau)$	$G(f) * H(f) = \int G(\nu)H(f-\nu)d\nu$

Switch τ and f in the table of formulas gives the Fourier transform pair

$$\begin{cases} 1-\alpha\lvert f\rvert & \text{if} \quad \lvert f\rvert \leq 1/\alpha \\ 0 & \text{if} \quad \lvert f\rvert > 1/\alpha \end{cases} \quad \Bigg| \quad \begin{cases} 1/\alpha & \text{if} \quad \tau = 0 \\ \frac{2\alpha}{(2\pi\tau)^2}\left(1-\cos\left(\frac{2\pi\tau}{\alpha}\right)\right) & \text{if} \quad \tau \neq 0 \end{cases}$$

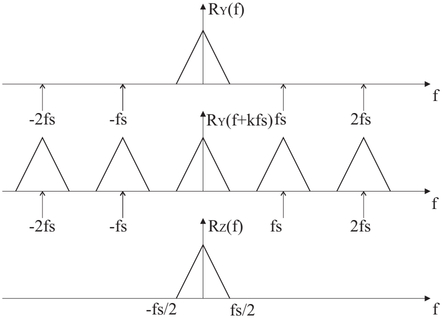
With $\alpha = 1$ we recognize $R_Y(f)$ and the covariance function will accordingly be

$$r_Y(\tau) = \begin{cases} 1 & \text{if} \quad \tau = 0 \\ \frac{2}{(2\pi\tau)^2}\left(1-\cos(2\pi\tau)\right) & \text{if} \quad \tau \neq 0 \end{cases}$$

b) Solution

We use $R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + 3k)$, $-3/2 < f \leq 3/2$, where the only contribution inside the limits is

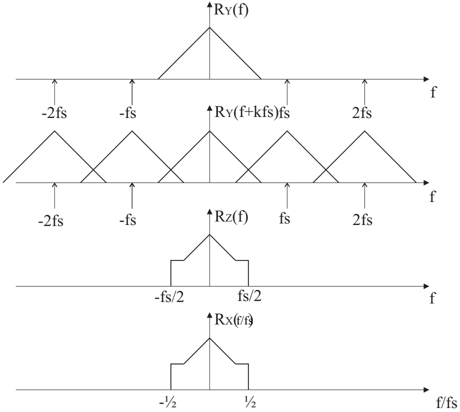
$$R_Z(f) = R_Y(f) \quad -3/2 < f \leq 3/2.$$



c) Solution

We use $R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + \frac{3}{2}k)$, $-3/4 < f \leq 3/4$, where the contributions inside the limits now are

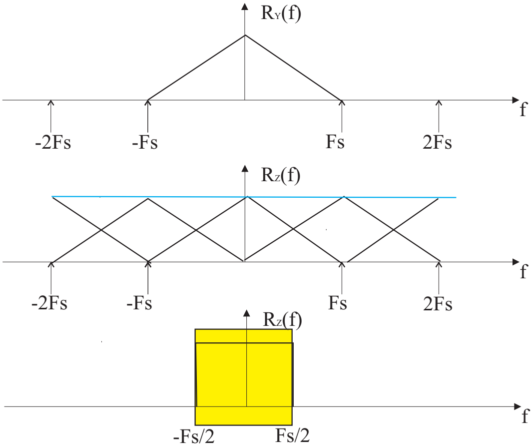
$$R_Z(f) = R_Y(f) + R_Y(f - \frac{3}{2}) + R_Y(f + \frac{3}{2}) \quad -3/4 < f \leq 3/4.$$



d) Solution

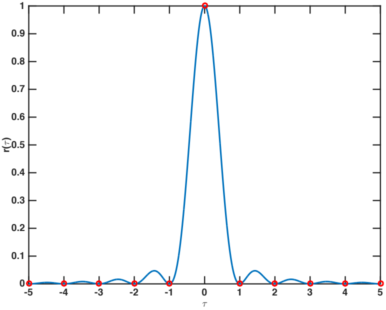
With $f_s = 1$ the aliasing turns the sampled process into a white noise process, where the contributions inside the limits are

$$R_Z(f) = R_Y(f) + R_Y(f-1) + R_Y(f+1) \quad -1/2 < f \leq 1/2.$$



d) Solution

The white noise covariance is also clearly shown if the covariance function $r_Y(\tau)$, $\tau \in \mathbb{R}$ (blue line) is sampled with $d = 1$ giving $r_Z(\tau) = 1$, for $\tau = 0$ and $r_Z(\tau) = 0$ for $\tau = \pm 1, \pm 2, \dots$ (red circles).



Answer

a) The covariance function is

$$r_Y(\tau) = \begin{cases} 1 & \tau = 0 \\ \frac{2}{(2\pi\tau)^2}(1 - \cos(2\pi\tau)) & \tau \neq 0. \end{cases}$$

b) The spectral density is

$$R_Z(f) = \begin{cases} 1 - |f| & |f| \leq 1 \\ 0 & 1 < |f| \leq \frac{3}{2}. \end{cases}$$

c) The spectral density is

$$R_Z(f) = \begin{cases} 1 - |f| & |f| \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < |f| \leq \frac{3}{4}. \end{cases}$$

d) The spectral density is

$$R_Z(f) = 1 \quad |f| \leq \frac{1}{2}$$

and the covariance function

$$r_Z(\tau) = \begin{cases} 1 & \tau = 0 \\ 0 & \tau = \pm 1, \pm 2, \ldots . \end{cases}$$