Sampling exercise

A stationary continuous time process $Y(t), t \in \mathbb{R}$, has the spectral density

$$R_Y(f) = \begin{cases} 
1 - |f| & |f| \leq 1 \\
0 & |f| > 1.
\end{cases}$$

\(a\) Determine the covariance function of the process.

\(b\) The process $Y(t)$ is sampled with $f_s = 3$, giving a the discrete time process $Z_t, t = 0 \pm d, \pm 2d, \ldots$. Determine the spectral density for the sampled process $Z_t$.

\(c\) The process $Y(t)$ is now sampled with $f_s = 3/2$ and another discrete time process $Z_t, t = 0 \pm d, \pm 2d, \ldots$, is received. Determine the spectral density for the sampled process $Z_t$.

\(d\) Finally, if instead $Y(t)$ is sampled with $f_s = 1$, determine the spectral density \textbf{and the covariance function} of the sampled process $Z_t, t = 0 \pm d, \pm 2d, \ldots$. 
a) Solution

Switch $\tau$ and $f$ in the table of formulas gives the Fourier transform pair

\[
\begin{align*}
\{ & \frac{\sin(\alpha \tau)}{2\pi \alpha} & \text{if } \tau \neq 0 \\
& 1 - \alpha |\tau| & \text{if } |\tau| \leq \frac{1}{\alpha} \\
& 0 & \text{if } |\tau| > \frac{1}{\alpha} \}
\end{align*}
\begin{align*}
\{ & 0 & \text{if } |f| > \frac{1}{\alpha} \\
& \frac{1}{\alpha} & \text{if } f = 0 \\
& \frac{2\alpha}{(2\pi \alpha)^2} \left(1 - \cos \left(\frac{2\pi f}{\alpha}\right)\right) & \text{if } f \neq 0
\end{align*}
\]

$g(\tau)h(\tau)$

$G(f) \ast H(f) = \int G(\nu)H(f - \nu)d\nu$

With $\alpha = 1$ we recognize $R_Y(f)$ and the covariance function will accordingly be

\[
r_Y(\tau) = \begin{cases} 
1 & \text{if } \tau = 0 \\
\frac{2}{(2\pi \tau)^2} \left(1 - \cos \left(\frac{2\pi \tau}{\alpha}\right)\right) & \text{if } \tau \neq 0
\end{cases}
\]
b) Solution

We use $R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + 3k)$, $-3/2 < f \leq 3/2$, where the only contribution inside the limits is

$$R_Z(f) = R_Y(f) \quad -3/2 < f \leq 3/2.$$
c) Solution

We use $R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + \frac{3}{2} k)$, $-3/4 < f \leq 3/4$, where the contributions inside the limits now are

$$R_Z(f) = R_Y(f) + R_Y(f - \frac{3}{2}) + R_Y(f + \frac{3}{2}) \quad -3/4 < f \leq 3/4.$$
d) Solution

With $f_s = 1$ the aliasing turns the sampled process into a white noise process, where the contributions inside the limits are

$$R_Z(f) = R_Y(f) + R_Y(f - 1) + R_Y(f + 1) \quad -1/2 < f \leq 1/2.$$
d) Solution

The white noise covariance is also clearly shown if the covariance function $r_Y(\tau), \tau \in \mathbb{R}$ (blue line) is sampled with $d = 1$ giving $r_Z(\tau) = 1$, for $\tau = 0$ and $r_Z(\tau) = 0$ for $\tau = \pm 1, \pm 2, \ldots$ (red circles).
a) The covariance function is

\[ r_{Y}(\tau) = \begin{cases} 
1 & \tau = 0 \\
\frac{2}{(2\pi \tau)^2} (1 - \cos(2\pi \tau)) & \tau \neq 0.
\end{cases} \]

b) The spectral density is

\[ R_{Z}(f) = \begin{cases} 
1 - |f| & |f| \leq 1 \\
0 & 1 < |f| \leq \frac{3}{2}.
\end{cases} \]

c) The spectral density is

\[ R_{Z}(f) = \begin{cases} 
1 - |f| & |f| \leq \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} < |f| \leq \frac{3}{4}.
\end{cases} \]

d) The spectral density is

\[ R_{Z}(f) = 1 \quad |f| \leq \frac{1}{2} \]

and the covariance function

\[ r_{Z}(\tau) = \begin{cases} 
1 & \tau = 0 \\
0 & \tau = \pm 1, \pm 2, \ldots.
\end{cases} \]