Example 2.5 (modified)

From a stationary stochastic process $U_t$, for $t = 0, \pm 1, \pm 2, \ldots$ of independent variables we construct a new stationary stochastic process $X_t$ by

$$X_t = U_t - 0.5U_{t-1}, \quad t = 0, \pm 1, \pm 2, \ldots$$

Calculate the mean value $m_X$ and covariance function $r_X(\tau)$ for $\tau = 0, \pm 1, \pm 2, \ldots$ if $U_t$ has the mean value, $m_U = m$ and variance $r_U(0) = \sigma^2$. 
Example 2.5

A stationary stochastic process $U_t$ of independent variables is defined by a variance $V[U_t] = C[U_t, U_t] = r_U(0)$ and all covariances

$$r_U(\tau) = C[U_{t-\tau}, U_t] = 0, \quad \tau = \pm 1, \pm 2, \ldots.$$ 

Another name is a white noise process.
Example 2.5

The mean value of $X_t$, $t = 0, \pm 1, \pm 2, \ldots$ is

$$m_X = E[X_t] = E[U_t - 0.5U_{t-1}] = E[U_t] - 0.5E[U_{t-1}],$$

using the rules for expected value of stochastic variables (see the formula booklet). With $m_U = E[U_t] = m$ for all values of $t$

$$m_X = m_U - 0.5m_U = 0.5m.$$
Example 2.5

The variance of $X_t$, $t = 0, \pm 1, \pm 2, \ldots$ will become

$$r_X(0) = V[X_t] = C[X_t, X_t] =$$
$$= C[U_t - 0.5U_{t-1}, U_t - 0.5U_{t-1}] =$$
$$= V[U_t] - 0.5C[U_t, U_{t-1}] - 0.5C[U_{t-1}, U_t] + 0.25V[U_{t-1}],$$

using the rules for variance of stochastic variables. We identify

$$V[U_t] = V[U_{t-1}] = r_U(0),$$

and

$$C[U_t, U_{t-1}] = r_U(-1), \quad C[U_{t-1}, U_t] = r_U(1).$$

For the white noise process, (independent stochastic variables), $r_U(1) = r_U(-1) = 0$. From this follows

$$r_X(0) = r_U(0) + 0.25r_U(0) = 1.25r_U(0) = 1.25\sigma^2.$$
Example 2.5

The covariances of \( X_t \) will be as follows:

\[
\begin{align*}
\rho_X(1) &= C[X_{t-1}, X_t] = C[U_{t-1} - 0.5U_{t-2}, U_t - 0.5U_{t-1}] = \\
&= C[U_{t-1}, U_t] - 0.5V[U_{t-1}] - 0.5C[U_{t-2}, U_t] + 0.25C[U_{t-2}, U_{t-1}] = \\
&= \rho_U(1) - 0.5\rho_U(0) - 0.5\rho_U(2) + 0.25\rho_U(1) = -0.5\rho_U(0) = -0.5\sigma^2,
\end{align*}
\]

where most terms are zero according to definition of \( \rho_U(\tau) \). Similarly we find

\[
\begin{align*}
\rho_X(2) &= C[X_{t-2}, X_t] = C[U_{t-2} - 0.5U_{t-3}, U_t - 0.5U_{t-1}] = \\
&= C[U_{t-2}, U_t] - 0.5C[U_{t-2}, U_{t-1}] - 0.5C[U_{t-3}, U_t] + 0.25C[U_{t-3}, U_{t-1}] = \\
&= \rho_U(2) - 0.5\rho_U(1) - 0.5\rho_U(3) + 0.25\rho_U(2) = 0.
\end{align*}
\]

Following the same procedure for other values of \( \tau \) we find

\[
\rho_X(\tau) = 0, \quad \tau \geq 2.
\]
Example 2.5

Next we can use the general rule for real-valued stochastic variables:

\[ r_X(-\tau) = C[X_t, X_{t-\tau}] = E[X_tX_{t-\tau}] - m_X^2 = E[X_{t-\tau}X_t] - m_X^2 = C[X_{t-\tau}, X_t] = r_X(\tau). \]

We therefore easily find

\[ r_X(-1) = -0.5\sigma^2, \]

and

\[ r_X(\tau) = 0, \quad |\tau| \geq 2. \]
Example 2.5-Answer

The expected value is

\[ m_X = 0.5m, \]

and the covariance function

\[
 r_X(\tau) = \begin{cases} 
 1.25\sigma^2, & \tau = 0 \\
 -0.5\sigma^2, & \tau = \pm 1 \\
 0, & |\tau| \geq 2.
\end{cases}
\]