SCHATTEN-VON NEUMANN PROPERTIES OF WEYL OPERATORS OF HÖRMANDER TYPE

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Let \( t \in \mathbb{R} \) be fixed and consider the pseudo-differential operators \( \text{Op}_t(a) \) with symbol \( a \) which is defined by the formula:

\[
\text{Op}_t(a)f(x) \equiv (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} a((1-t)x + ty, \xi) e^{i(x-y, \xi)} dyd\xi
\]

It is well-known that if \( 0 \leq \delta < \rho \leq 1 \) and \( r \in \mathbb{R} \), then each \( \text{Op}_t(a) \) with \( a \in \text{S}^r_{\rho, \delta}(\mathbb{R}^{2n}) \) is \( L^2 \)-continuous, if and only if \( \text{S}^r_{\rho, \delta}(\mathbb{R}^{2n}) \subseteq L^\infty \) (i.e. \( r \leq 0 \)). Here \( \text{S}^r_{\rho, \delta}(\mathbb{R}^{2n}) \) consists of all \( a \in C^\infty(\mathbb{R}^{2n}) \) such that

\[
|\partial_\alpha x \partial_\beta \xi a(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{r-\rho|\beta|+\delta|\alpha|}.
\]

More recently, results which are focused on "individual symbols" instead of whole symbol classes can be found in e.g. [1], from which it follows that if \( a \in \text{S}^r_{\rho, \delta} \) for some \( r \), then \( \text{Op}_t(a) \) is \( L^2 \)-continuous, if and only if \( a \in L^\infty \).

The general theory involving these results, is formulated within the Hörmander-Weyl calculus, where the symbol classes \( \text{S}(m, g) \) are parameterized with weight functions \( m \) and Riemannian metrics \( g \). The continuity investigations also involve Schatten properties. Especially, the following general result is deduced: Let \( p \in [1, \infty] \), and assume that the \( g \)-Planck’s function \( h_g \) satisfies \( h_g^N m \in L^p \), for some \( N \geq 0 \). Then \( \text{Op}_t(a) \) is a Schatten-\( p \) operator, if and only if \( a \in L^p \).

Recently, a related result was obtained also when \( p \leq 1 \). More precisely, in [2] it is proved that if \( p \in (0, 1] \), \( m \in L^p \) and \( a \in \text{S}(m, g) \), then \( \text{Op}_t(a) \) is a Schatten-von Neumann operator of order \( p \).

In the talk we explain these results with explicit examples, and present some ideas of some proofs.


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