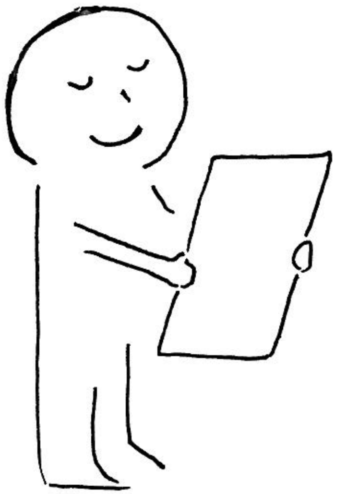


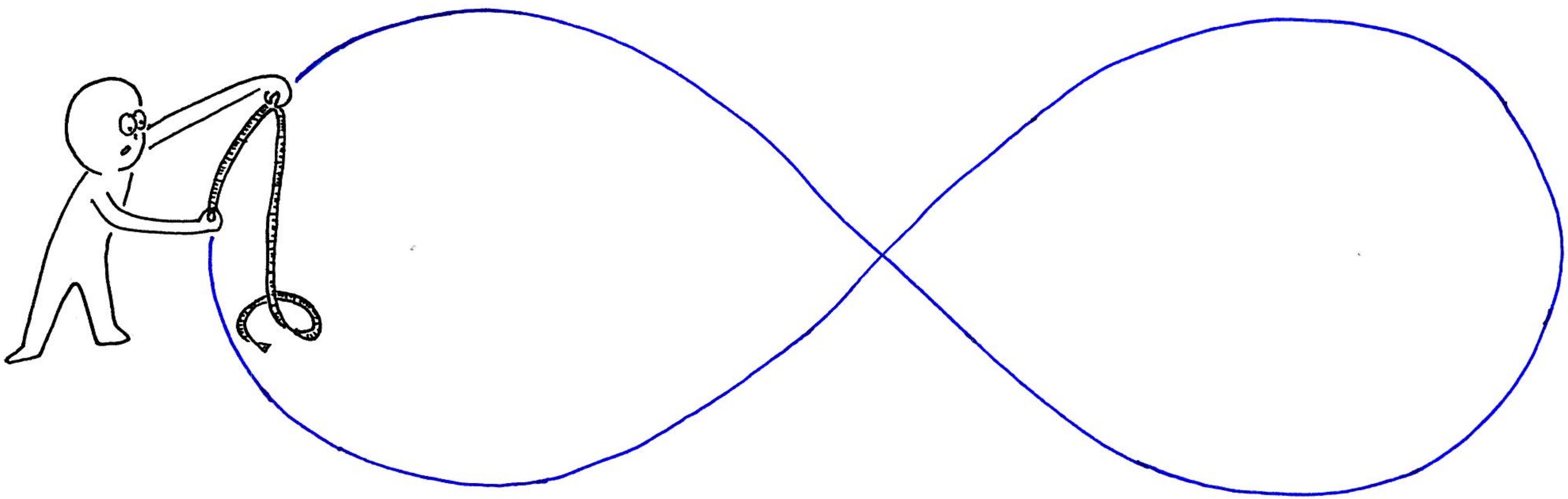
Means means means

Amol Sasane

3 means / averages

- Arithmetic mean
- Geometric mean
- Arithmetic - geometric mean

Gauss : Length of the 'lemniscate'.

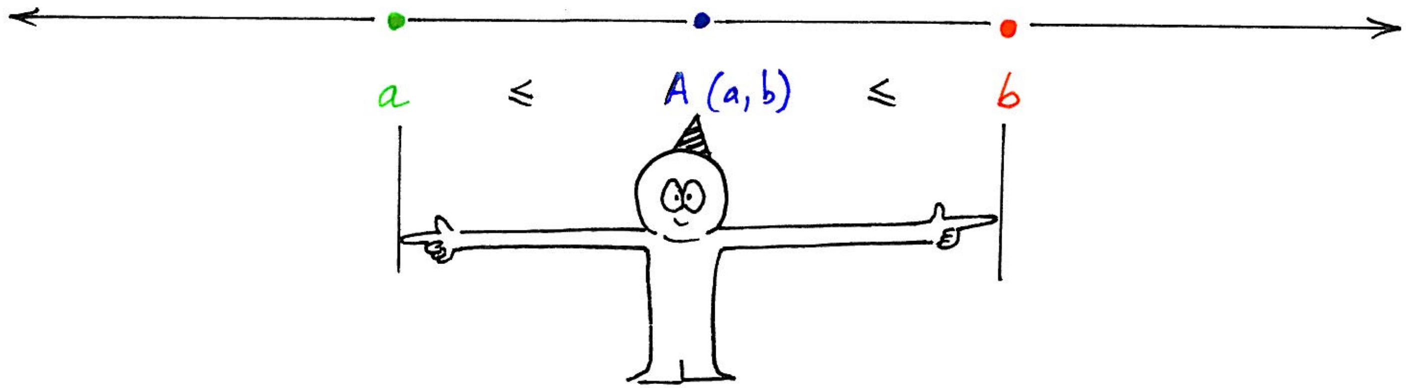


$$0 < a \leqslant b$$

mean/average

(1) Arithmetic mean $\quad A(a,b) := \dfrac{a+b}{2}$

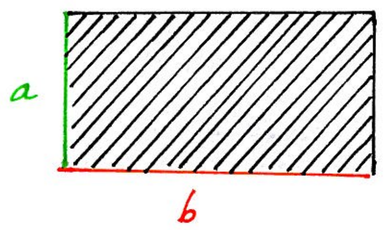
$$A(1,5) = \frac{1+5}{2} = \frac{6}{2} = 3$$



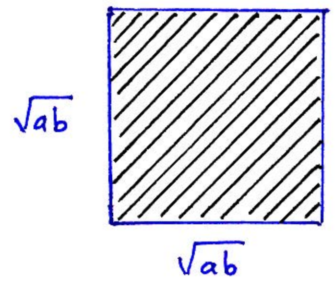
(2) Geometric mean $\qquad G(a,b) := \sqrt{ab}$

"Geometric"

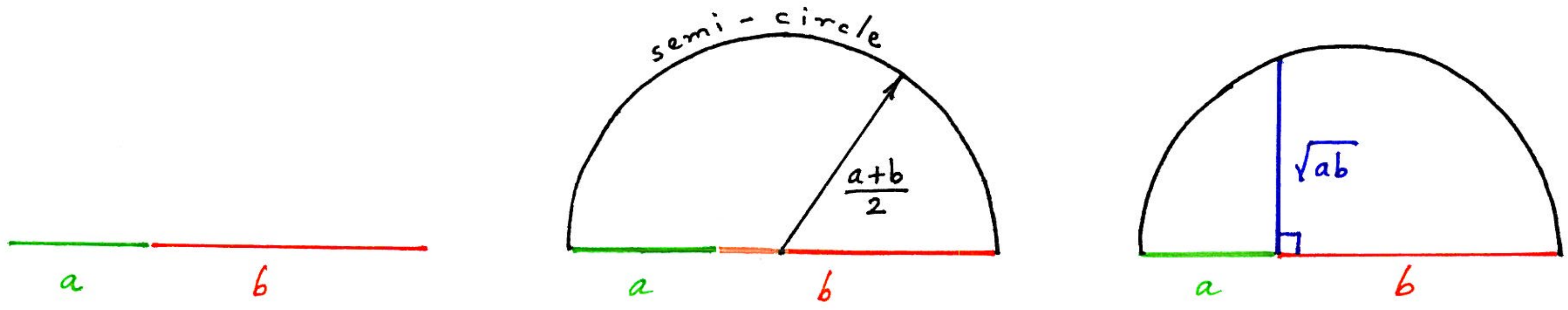
(a) Squaring the rectangle:

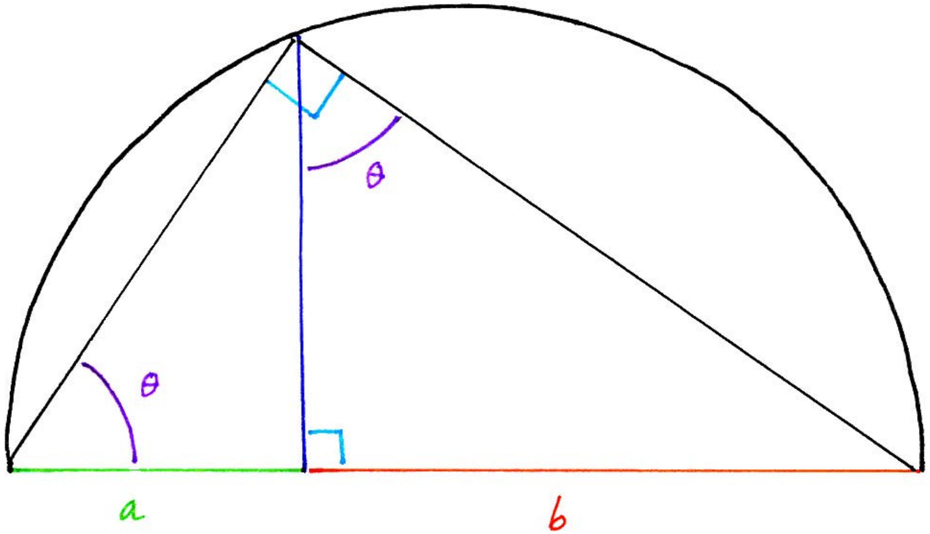


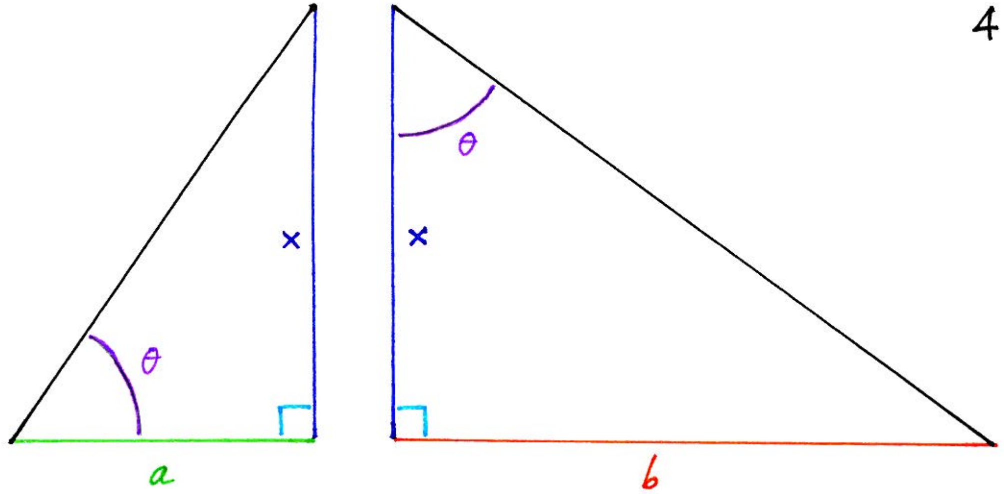
$=$ (Area)



(b) Construction of $G(a,b)$:

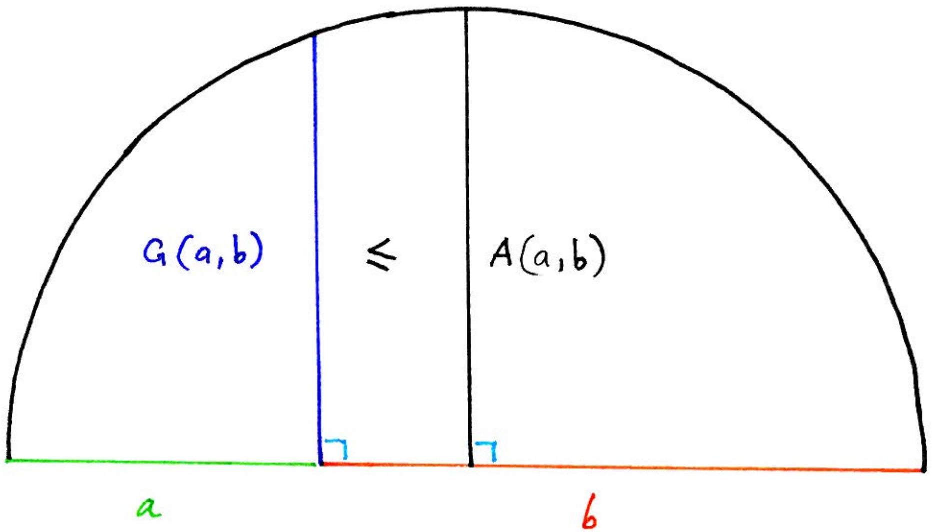






$$\frac{a}{x} = \frac{x}{b} \qquad x^2 = ab \qquad x = \sqrt{ab}$$

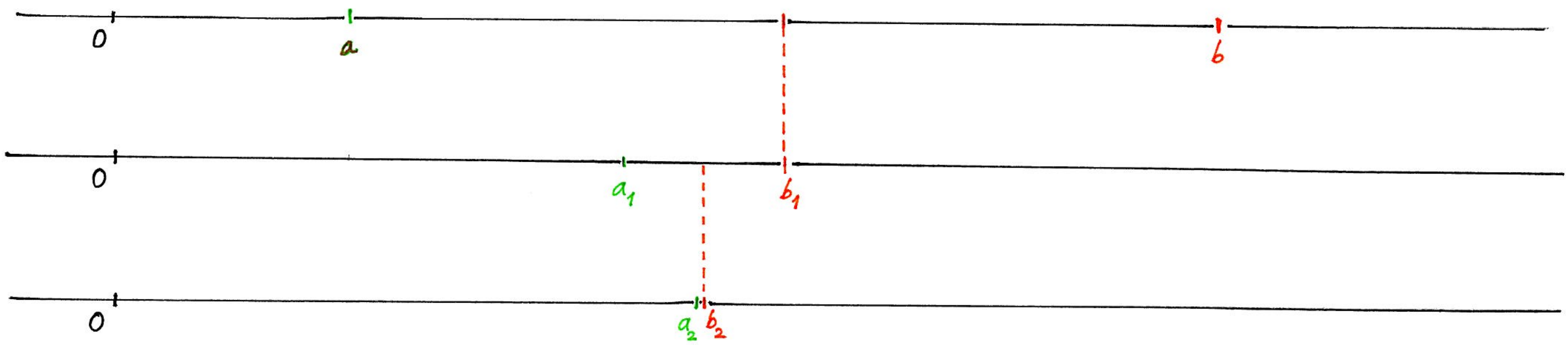
□



Arithmetic mean - Geometric mean inequality:

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

(3) The arithmetic - geometric mean

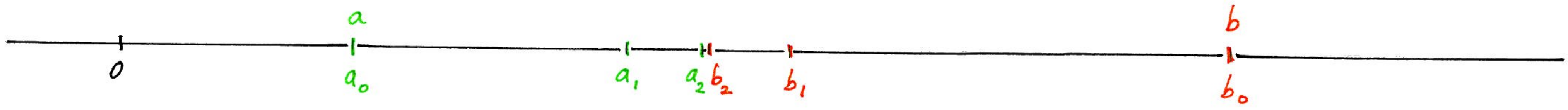


Input: $a < b$
$(a_0) \quad (b_0)$

Calculate: $a_{n+1} := \sqrt{a_n b_n} < b_{n+1} := \dfrac{a_n + b_n}{2}$, $n = 0, 1, 2, 3, \ldots$

Then:

$$\boxed{AG(a, b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.}$$



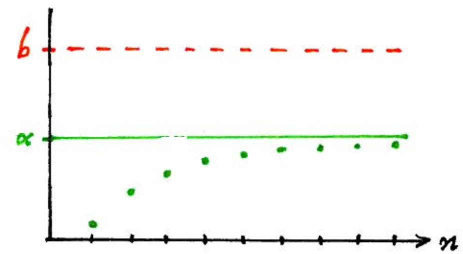
$$a_{n+1} := \sqrt{a_n b_n} < b_{n+1} := \frac{a_n + b_n}{2}, \qquad n = 0, 1, 2, 3, \dots .$$

Theorem.

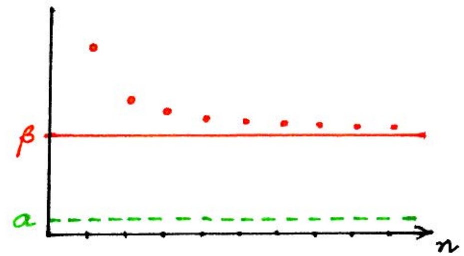
$$\boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =: AG(a, b).}$$

Proof

$$a_0 < a_1 < a_2 < a_3 < \dots < b$$



$$a < \quad \dots < b_3 < b_2 < b_1 < b_0$$



$$b_{n+1} = \frac{a_n + b_n}{2} \xRightarrow{n\to\infty} \beta = \frac{\alpha + \beta}{2} \Longrightarrow \alpha = \beta . \qquad \square$$

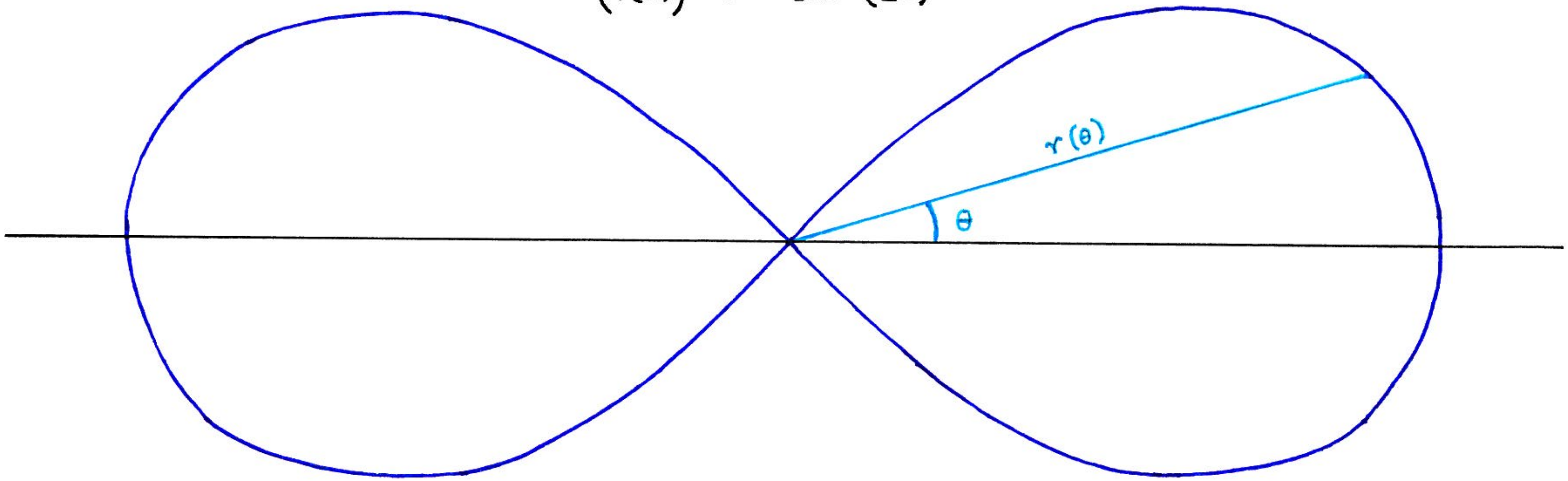
n	a_n	b_n
0	1. 000000 000 000 000 000000	1. 414 21356 2373 905 0 48802
1	1. 189 207 115 002 721 066 717	1. 207 106 781 18654752 4401
2	1. 198 1235 21 493 120 112 607	1.19 81 5694 8094 6342 95559
3	1. 198 140 234 677 307 205 798	1.1981 4023 4793 87720 9083
4	1. 19814023 4735 5922 07439	1. 1981 4023 4735 5922 07441

De origine proprietatibusque numeroru, arithmetico-geometricorum

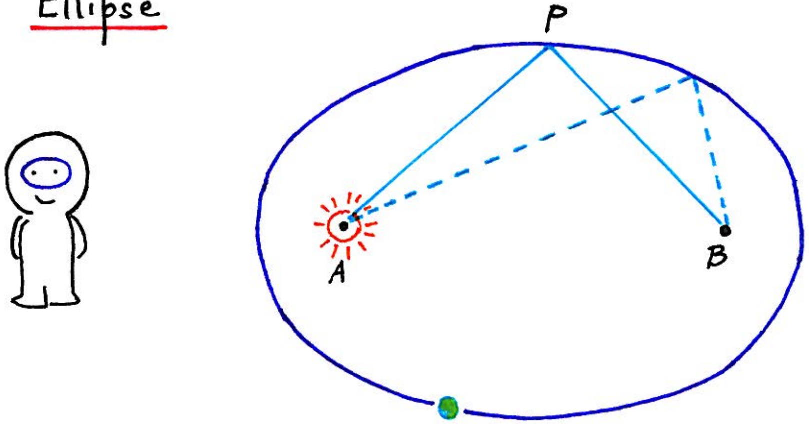
1800

The lemniscate

$$(r(\theta))^2 = \cos(2\theta)$$

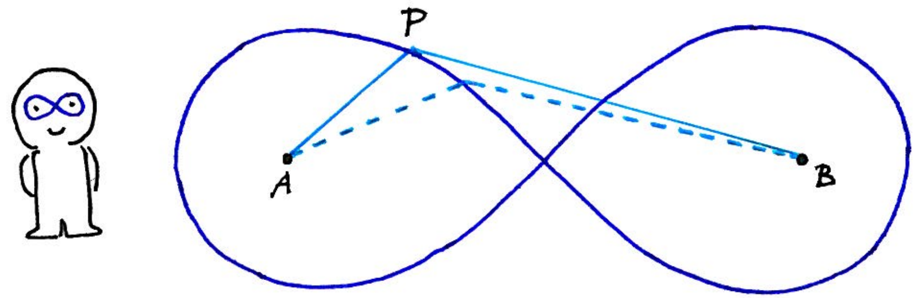


Ellipse



$$PA + PB = \text{constant}$$

Lemniscate



$$PA \cdot PB = \text{constant}.$$

Length of the lemniscate

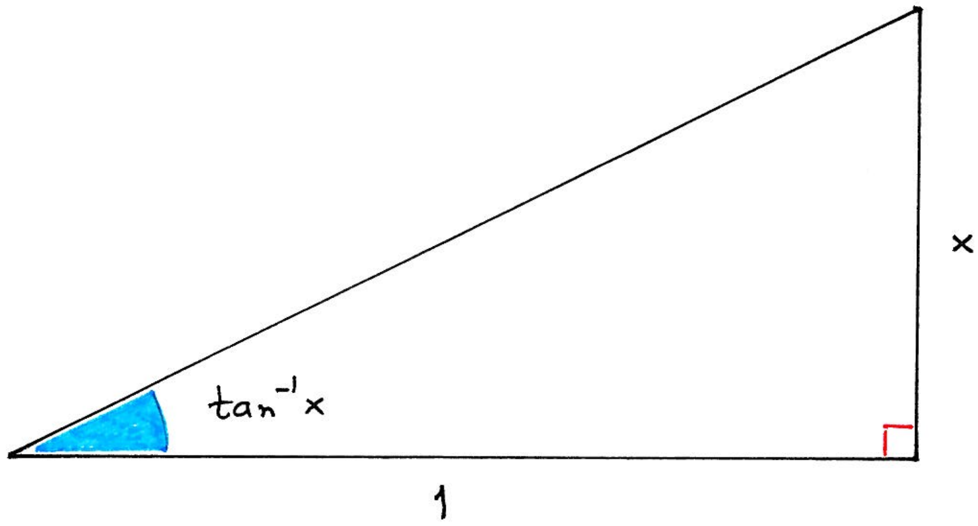
$$= 2 \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+1)(x^2+2)}}$$

Not expressible in terms of elementary functions!

Gauss

1777 – 1855

arctan or $\tan^{-1}$

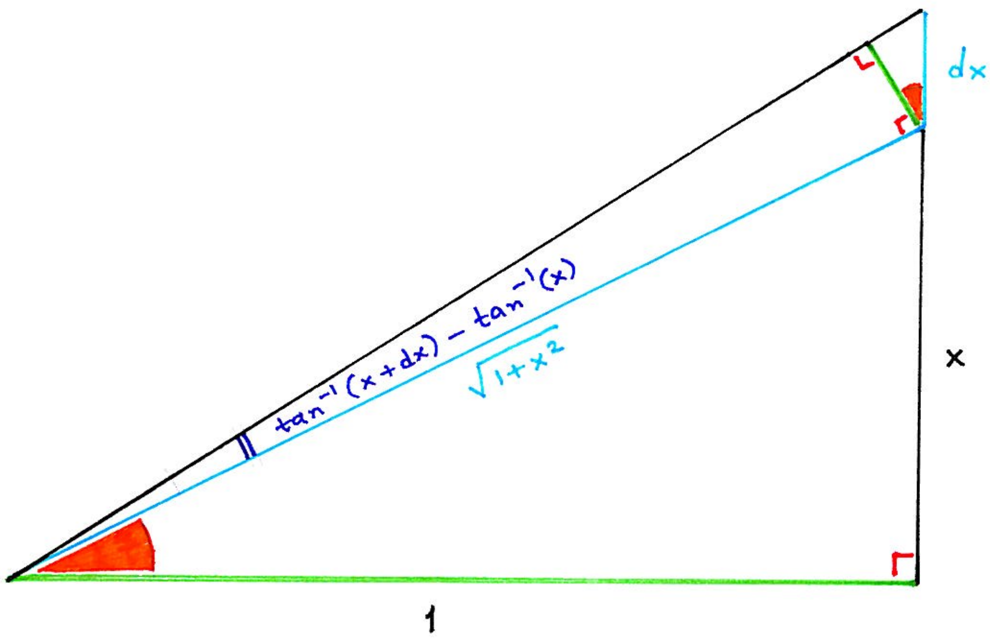


What is

$$\frac{d}{dx} \tan^{-1} x \quad ?$$

$$\lim_{dx \to 0} \frac{\tan^{-1}(x + dx) - \tan^{-1} x}{dx}$$

$$\lim_{dx \to 0} \frac{\tan^{-1}(x+dx) - \tan^{-1} x}{dx}$$



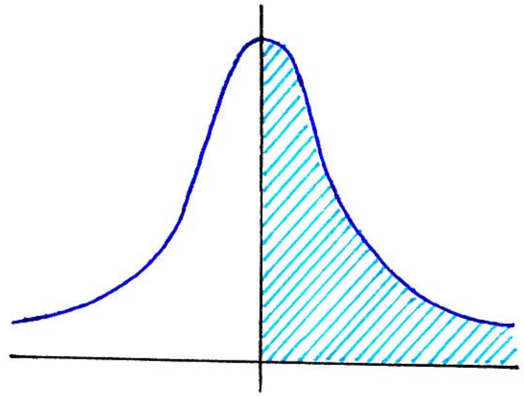
$$\frac{\text{side}}{\text{hypotenuse}} = \frac{\left(\tan^{-1}(x+dx) - \tan^{-1} x\right)\sqrt{1+x^2}}{dx} = \frac{1}{\sqrt{1+x^2}} = \frac{\text{side}}{\text{hypotenuse}} \Rightarrow \frac{d\tan^{-1} x}{dx} = \frac{1}{1+x^2}.$$

<u>Corollary</u>.

$$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{x^2+a^2}.$$

$$\left(\frac{1}{a} \cdot \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a} = \frac{1}{x^2+a^2}\right)$$

Our theorem: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}.$

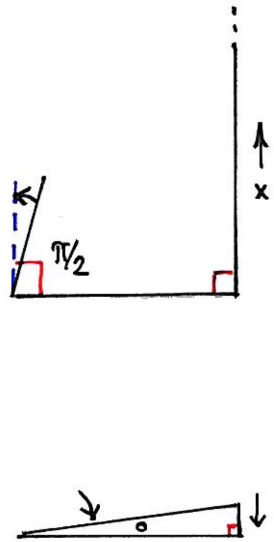


Proof.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = 2 \int_{0}^{\infty} \frac{dx}{x^2+a^2}$$

$$= 2 \int_{0}^{\infty} \frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) dx$$

$$= 2 \cdot \frac{1}{a} \cdot \left(\tan^{-1}\infty - \tan^{-1} 0\right)$$



$$= 2 \cdot \frac{1}{a} \cdot \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{a}.$$

$\square$

Our theorem:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{a}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+a^2)}} = \frac{\pi}{a}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+b^2)(x^2+b^2)}} = \frac{\pi}{b} .$$

But what about:

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{\pi}{\boxed{?}}$$

$$a \leqslant \boxed{?} \leqslant b$$

Theorem (Gauss): $$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{\pi}{AG(a,b)}.$$

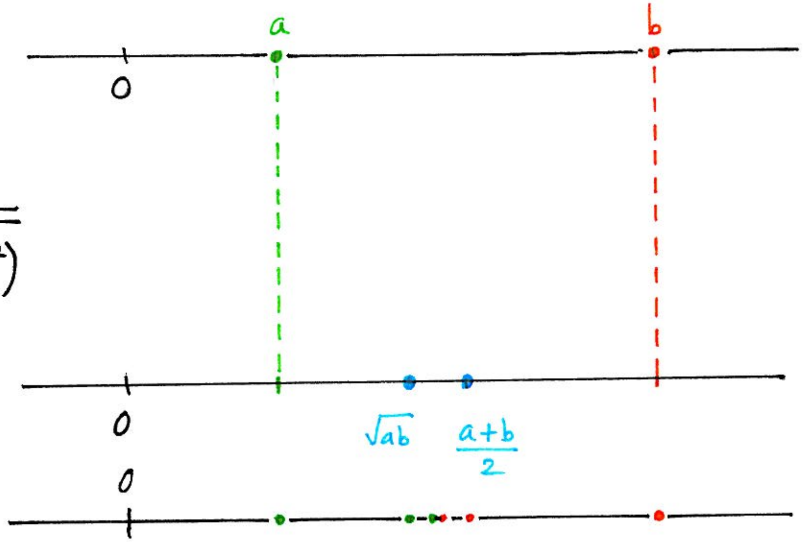
Proof.

$$I(a,b) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}$$

$$\left(t = x - \frac{ab}{x}\right)$$

$$= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{\left(x^2+\sqrt{ab}^2\right)\left(x^2+\left(\frac{a+b}{2}\right)^2\right)}}$$

$$= I\left(\sqrt{ab}, \frac{a+b}{2}\right)$$



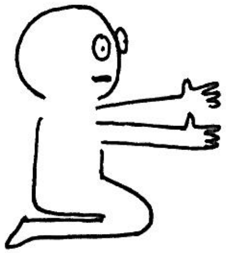
So:

$$I(a,b) = I(a_1,b_1) = I(a_2,b_2) = I(a_3,b_3) = \cdots$$

$$= \lim_{n\to\infty} I(a_n,b_n) = I\left(AG(a,b), AG(a,b)\right)$$

$$= \int_{-\infty}^{\infty} \frac{dx}{x^2 + AG(a,b)^2} = \frac{\pi}{AG(a,b)}.$$

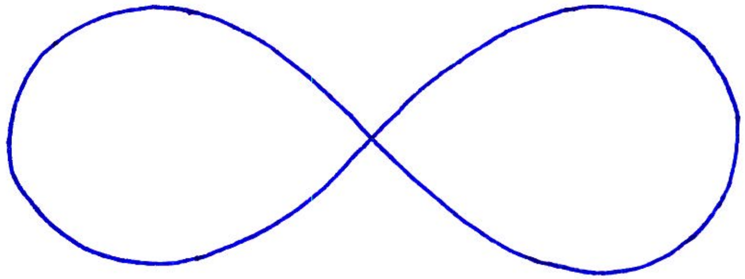
$\square$



Theorem

(Gauss, 1799, age < 22)

Length of



is $\dfrac{2\pi}{AG(1, \sqrt{2})}$.