

Computer exercise 3 in Stationary stochastic processes

The purpose of this exercise is to study different techniques for spectral estimates and cross-spectrum estimates.

1 Preparations

- Download and read `Matlabhints`.
- Find out if you have access to the MATLAB toolboxes `filterdesign` and `ident`. If not, you have to skip some of the exercises.
- Study Chapter 6.5 and 8 in the textbook.
- Read through the entire computer exercise, paying extra attention to section 2.1.
- Answer the exercises in the question dictionary below (use additional sheets, if needed); you are expected to be able to explain your answers during the exercise.
- In exercise 7, you are expected to use MATLAB with less detailed instructions. Therefore, you are expected to familiarize yourself with relevant functions before the exercise. If you don't feel at ease with MATLAB, it is wise to start working on this exercise before the lab session.

1.1 Question dictionary

1. Briefly explain how you can improve the periodogram estimate using a windowing technique. What are the advantages and disadvantages?

Your answer:

2. How should one interpret a cross spectrum?

Your answer:

3. What is the connection between the spectral densities for input and output processes in a linear filter?

Your answer:

4. Define the quadratic coherence spectrum. What is the minimum and maximum value of the function?

Your answer:

5. The cross spectrum of the input process $X(t)$ and the output process $Y(t)$ from the filter with frequency function $H(f)$ is

$$R_{X,Y}(f) = H(f)R_X(f).$$

For a white noise input, $R_X(f) = 1$ and $R_{X,Y}(f) = H(f)$. What is the quadratic coherence spectrum $\kappa_{X,Y}(f)$?

Your answer:

6. Let an uncorrelated disturbance process $Z(t)$ be added to the filtered process, i.e., $Y(t) = \int h(t-u)X(u) du + Z(t)$, where $X(t)$ is white noise. What is $R_{X,Y}(f)$ and $\kappa_{X,Y}(f)$ in this case?

Your answer:

2 ARMA processes in MATLAB

This section constitutes a brief introduction to spectral analysis and simulation of ARMA processes in MATLAB. It is intended for study and experimentation before the lab session, and as a reference during the exercise.

2.1 Spectral density and covariance function of ARMA models

Initiate the row vectors of the ARMA(4,2)-model

$$X(t) - 0.9X(t-1) + 0.8X(t-2) - X(t-3) + 0.8X(t-4) = e(t) + 0.4e(t-1) + 0.5e(t-2)$$

by writing in MATLAB

```
>> A = [1 -0.9 0.8 -1 0.8];  
>> C = [1 0.4 0.5];
```

By armagui

The simplest way to compute covariance function and spectral density for an ARMA-process is to use the routine `armagui`, which we studied in Computer exercise 2. Open `armagui` and introduce zeros and poles in the graphics window. Then export the result to `data` and check the details in the command window. There you can change the A- and C-coefficients to a new model end import to `armagui`.

By freqz in the Filter design toolbox

Another approach to compute spectral density for an ARMA-process is to use the commands

```
>> [H,w]=freqz(C,A);  
>> R=abs(H).^2;
```

where `freqz` is a MATLAB function in the Filter design toolbox that calculates the frequency function corresponding to the polynomials C and A in a number of equally spaced frequency values (default 512) between $0 \leq \omega < \pi$ where $\omega = 2\pi f$. The squared absolute value gives us the spectral density S . Visualize using

```
>> plot(w,R)
```

The command `semilogy(w,R)` gives a logarithmic scale in y-axis.

To calculate the covariance function the inverse Fourier transform of the spectral density is computed. The frequency range must be the whole period, i.e., $-0.5 \leq f < 0.5$ alternatively $0 \leq f < 1$ when the command `ifft` is used. The frequency function is computed for this range using `whole` in the function `freqz`,

```
>> H=freqz(C,A,512,'whole');
>> Rd=abs(H).^2;
>> r=ifft(Rd);
>> stem([0:127],r(1:128),'filled')
```

where the 128 first values of the covariance function is plotted. Observe that the vector element `r(1)` is equal to the covariance value at zero, i.e. $r_X(0) = r(1)$, and more generally $r_X(k) = r(k+1)$ in MATLAB. If you instead are interested in the correlation function $\rho(\tau) = r(\tau)/r(0)$ you obtain it by writing

```
>> stem([0:127], r(1:128)/r(1),'filled')
```

2.2 Simulation of ARMA-processes

To be able to simulate an ARMA-model one need a sequence of independent normally¹ distributed $\{e(t)\}$ random variables. Generate such a sequence with variance 1 by

```
>> m = 0;
>> sigma = 1;
>> e = normrnd(m, sigma, 1, n);
```

alternatively

```
>> e = randn(1,n);
```

Each call generate a row-matrix `e` with `n` normally random numbers, choose $n \approx 400$. The simulation is done with

```
>> x = filter(C, A, e);
```

which result in the vector `Y` with the simulated time-series as the result, where `A` and `C` are defined as earlier. If you want to plot the result you can write e.g.

```
>> plot(x)
```

¹In general $e(t)$ does not need to be normally distributed.

3 To start the MATLAB exercise

- Download the necessary m-files from the folder `ssproutines` and the data from `sspdata` and make them accessible from your MATLAB window.
- It is often convenient to gather MATLAB commands in a program script; the MATLAB editor can be used for this purpose. This is started with the command `edit` in the MATLAB window.

4 Investigation of different spectrum estimation techniques

We can use `armagui` to simulate different processes by first importing a model into `armagui` and then exporting the generated realisation back to MATLAB. The realization can then be analyzed in `spekgui` using the there available methods.

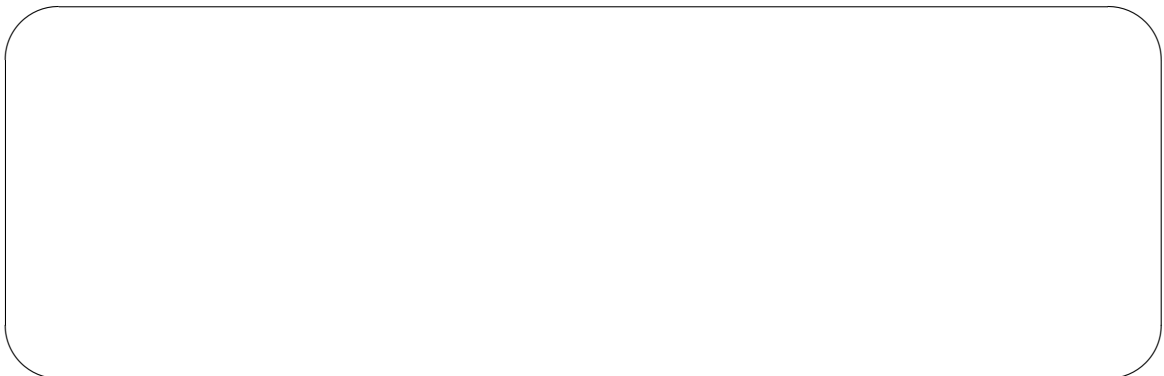
Simulate a 500 sample realization of an ARMA(4,2)-process.

```
>> modell.A=[1 -2.39 3.35 -2.34 0.96];  
>> modell.C=[1 0 1];
```

by importing the model to `armagui`.

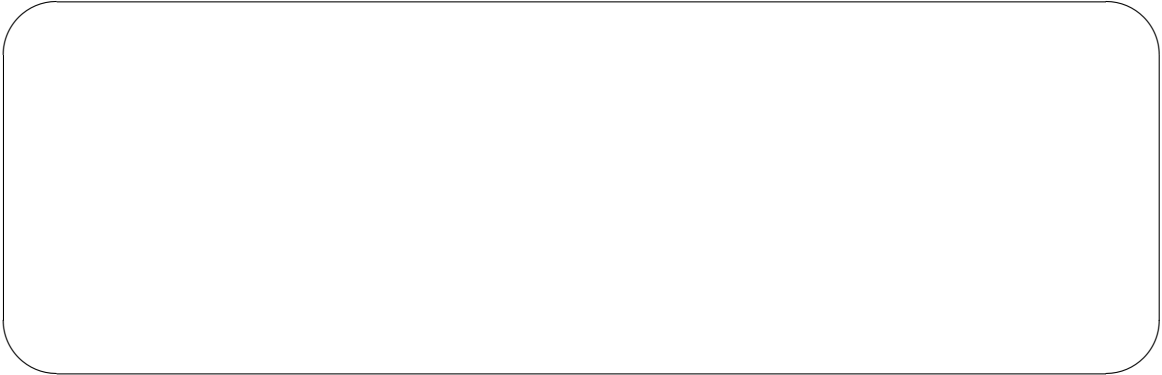
1. Initially, we examine the spectral density in linear scale. Generate and export a realisation of data to MATLAB and then import this variable into `spekgui`. Compare the periodogram estimate in linear scale with the true spectral density in `armagui`. Simulate a new realization in `armagui`, export back to MATLAB and then import it into `spekgui`. Repeat this to see the variance of the periodogram estimate.

Q. Draw rough sketches of periodogram estimates for two different realizations. Do the peak heights vary considerably?



2. Similarly, analyze some simulations using averaged periodogram method with 10 averages (parameter set to 10).

Q. Draw rough sketches of periodogram estimates for two different realizations. How many times lower variance should the averaged periodogram estimate have, according to theory?



3. Choose e.g., 2, 4, 10, 20, 30 as the parameter value for the averaged periodogram method.

Q. How is the spectrum estimate affected? What do we lose by increasing the number of averages?



4. Study the spectral density and the spectral estimates using a log-scale.

Q. Draw a rough sketch of the log-scale view of spectral density. How is the zero of the unit circle reflected in the spectral density? At what frequency does the zero occur?



5. Instead use Welch's method with a time-window and some different averages, e.g., 2, 4, 10.

Q. Is the spectral estimate improved? What is the drawback?

Assume now that we do not know that the observations are independent. From the compendium, we know that

$$V(m^*) \approx \frac{1}{n} \sum_{k=-n+1}^{n-1} r(k),$$

A natural estimate of $V(m^*)$ can be found by simply replacing $r(k)$ with $r^*(k)$:

$$\begin{aligned} V^*(m^*) &= \frac{1}{n} \sum_{k=-n+1}^{n-1} r^*(k) = \frac{1}{n} \left(r^*(0) + 2 \sum_{k=1}^{n-1} r^*(k) \right) \\ &= \frac{1}{n^2} \left(\sum_{j=1}^n (x_j - \bar{x})^2 + 2 \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} (x_k - \bar{x})(x_{k+j} - \bar{x}) \right) \\ &= \frac{1}{n^2} \left(\sum_{j=1}^n (x_j - \bar{x}) \right)^2 = 0. \end{aligned}$$

Note that this estimate of $V(m^*)$ is equal to zero, independently of the actual data set! This is clearly an unreasonable estimate. How do we solve this problem? As

$$\sum_{k=-\infty}^{\infty} r(k) = R(0),$$

one alternative estimate can be found as $V(m^*) = R(0)/n$.

5 Detection of flickering light in an EEG signal

The measurements of the electrical signals from the brain are called ElectroEncephaloGram (EEG) signals and are often clinically used for detection of illnesses as well as in cognitive experiments. The EEG signal is a challenging process to deal with as the information is usually heavily disturbed by the spontaneous EEG-activity. However, if applying a flickering light in front of a person (with closed eyes), this will generate an EEG signal containing this frequency, and in this case the response will be rather strong. In general, the EEG signal is a non-stationary process, but for shorter periods (seconds), the EEG activity can be assumed to be stationary. In this exercise, you will study a number of EEG sequences using `spekgui` and judge which sequence that is recorded from a measurement with a flickering light. Start by loading the file `eegdata12` which contains three different measurements of EEG (5 s) from the same person. One of the sequences are measured when a flickering light of 12 Hz was introduced.

1. Study the variables `data1`, `data2` and `data3` by loading them one by one into `spekgui` and examine the realization, the covariance estimate and the spectrum estimate. Which is best, the periodogram or the Welch method? Should the scale be linear or logarithmic? Can we gather any information from the covariance estimate? Which sequence is most likely to come from the 12 Hz flickering light?

Your Answer:

2. Load the file `eegdatax` and study the different variables `data1`, `data2` and `data3` in `spekgui`. Now the frequency of the flickering light is unknown. (Hint: it is between 5 and 20 Hz).

Q. Can you judge which frequency it has and which sequence it is introduced into?

6 Cross spectrum in input/output relations

The cross spectrum of the input signal $X(t)$ and the output signal $Y(t)$ from the filter with frequency function $H(f)$ is

$$R_{X,Y}(f) = H(f)R_X(f).$$

For a white noise input signal $R_X(f) = 1$ and $R_{X,Y}(f) = H(f)$. Use an ARMA(4,2)-process $X(t) - 0.9X(t-1) + 0.8X(t-2) - X(t-3) + 0.8X(t-4) = e(t) + 0.4e(t-1) + 0.5e(t-2)$

by writing

$$A = [1 \ -0.9 \ 0.8 \ -1 \ 0.8];$$

$$C = [1 \ 0.4 \ 0.5];$$

in MATLAB. Compute the filter for this ARMA-process, and a lowpass AR(1)-process as the uncorrelated disturbance $Z(t)$ i.e.,

```
[H, w]=freqz(C, A);
A2=[1 -0.95];
[H2, w]=freqz(1, A2);
kappa=abs(H).^2./(abs(H).^2+abs(H2).^2);
plot(w, kappa)
```


1. Change the location of the pole of the low-pass disturbance process, e.g.

$$A2 = [1 \ 0.95].$$

Q. What differs in the resulting coherence spectrum? Why?

2. Change the power of the disturbance process,

$$\text{kappa} = \text{abs}(H) .^2 ./ (\text{abs}(H) .^2 + \text{sigma}2 * \text{abs}(H2) .^2);$$

where $\text{sigma}2$ can be chosen as, e.g., 0.01, 0.1, 10 or 100.

Q. How does the choice affect the coherence spectrum? Can you intuitively explain this?

7 Analysis of EKG and index finger pulse data

Measurements of electrical signals in the heart are called ElectroCardioGram (EKG, from the german ElektroKardioGramm) signals. In medicine, this technique is commonly used to monitor the heart activity of a patient. Another way to indirectly monitor heart activity is to measure the pulse of the patient. In hospitals, this is commonly done using a small device on either index finger. In this exercise, we will analyze the correlation between the two, using the techniques studied in this course. In this exercise, you will be tasked with working out the details on how to treat the data yourself. However, helpful tips are provided.

Load the file `pulsedata`. The EKG data is contained in the vector called `ekg`, and the finger pulse measurements in `fingerPulse`. The sampling frequency is 1 kHz. Start by plotting the data, making sure the time axis is correct.

Q. Can you spot any periodicity? Ponder what similarities and differences you can discern between the two signals.

Estimation of covariance and spectrum

Estimate and plot the covariance functions and spectral densities of the two processes. The estimation can be done using the WAFO functions `dat2cov` and `dat2spec` available in `ssproutines`. These were used in Computer exercise 1. Remember that the input data is a matrix with time in first column and data in second. (To estimate spectrum you can also use `welch`.)

Make sure the frequency axis is scaled correctly, and note that we are primarily interested in low frequencies. Try plotting both using linear and logarithmic (`semilogy`) scale on the y axis.

Q. In which frequency range do you expect the heartbeat frequency, f_H , to be found? Can you find f_H in both spectra? In the spectra, there are prominent peaks at $2f_H, 3f_H, \dots$, can you explain why? In the measurements, there is interference at some specific frequencies (look at the logarithmic plot), which? Can you explain any of the interference peaks?

Estimation of cross-covariance and cross-spectrum

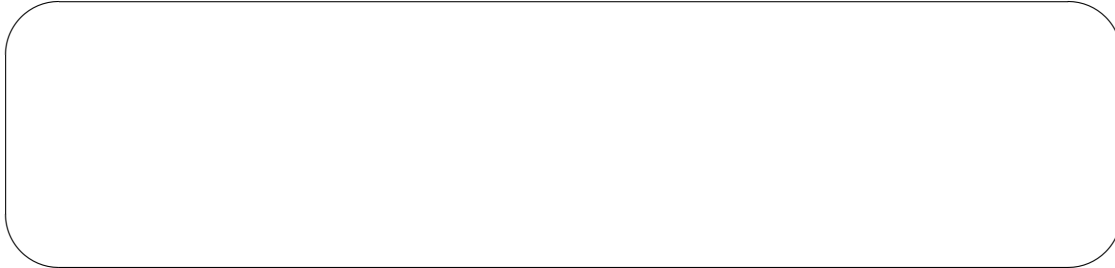
We now wish to estimate the cross-covariance function of the two processes. This is done using `xcov` from the MATLAB System identification toolbox, `ident`. Start by looking at the documentation for the function.

```
doc covf
```

Especially note the definition of cross-covariance used in MATLAB. Use the function to estimate and plot the cross-covariance function between the two processes up to a time lag

of a few seconds, with the time axis properly rescaled in seconds. Recall the differences between this function and the autocovariance function.

Q. How long is the delay from the electrical signal provoking the heartbeat until the blood rush reaches the finger?



Now, estimate the cross-spectral density of the two processes using `cpsd`. Recall that the cross-spectral density is a complex-valued function.

Q. How can we plot it to capture the relevant information? Can you still identify f_H ?



Finally, calculate and plot the estimated squared coherence spectrum between the two processes. Note that previous estimations can be used to calculate this.

Q. What is the coherence of the heartbeat frequency? What does this imply about how the blood pressure propagates through the body?

